

Spinor-Vector duality in heterotic string orbifolds

Carlo Angelantonj,^a Alon E. Faraggi^b and Mirian Tsulaia^{b,c}

^a*Dipartimento di Fisica Teorica and INFN Sezione di Torino,
Via P. Giuria 1, I-10125 Torino, Italy*

^b*Department of Mathematical Sciences, University of Liverpool,
Liverpool L69 7ZL, U.K.*

^c*Centre for Particle Physics and Cosmology, Ilia State University,
0162 Tbilisi, Georgia*

E-mail: carlo.angelantonj@to.infn.it, alon.faraggi@liv.ac.uk,
tsulaia@liv.ac.uk

ABSTRACT: The three generation heterotic-string models in the free fermionic formulation are among the most realistic string vacua constructed to date, which motivated their detailed investigation. The classification of free fermion heterotic string vacua has revealed a duality under the exchange of spinor and vector representations of the $SO(10)$ GUT symmetry over the space of models. We demonstrate the existence of the spinor-vector duality using orbifold techniques, and elaborate on the relation of these vacua to free fermionic models.

KEYWORDS: Superstrings and Heterotic Strings, Superstring Vacua

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1 Introduction

String theory provides a self-consistent framework to describe quantum gravity and particle physics in a unified way. Several approaches to particle phenomenology have been pursued, based on heterotic string compactifications, orientifold constructions, M-theory compactification on manifold of special holonomy and/or F-theory techniques. All these scenarios have brought new interesting ideas to particle physics and string theory, though none can be considered as “fully realistic”. Among the various approaches, heterotic string theory still seems to be a preferred candidate to build quasi-realistic models, and particularly promising is the free-fermionic construction of heterotic vacua [1, 2]. Although these constructions are typically formulated at special points in the moduli space and thus lack an apparent geometric description, over the last two decades they have shown to be very powerful tools to develop phenomenological string vacua [3–10]. Three generation models with the correct Standard Model charge assignments, as well as the canonical $SO(10)$ embedding of the weak hypercharge have been constructed, and various phenomenological issues have been further explored [11, 12].

More recently, classes of quasi-realistic heterotic string models have also been constructed, based on orbifold techniques [13–15], that also allow to explore the underlying moduli dependence of couplings and gauge groups. It should be stressed, however, that the two formulations — in terms of free fermions or in terms of free bosons — are closely related and the corresponding string vacua can be described equivalently using the two formalisms. Indeed, the free-fermionic constructions correspond in general to \mathbb{Z}_2^n toroidal orbifolds, when the geometric data of the six-torus are chosen to correspond to special points of moduli space.

It is therefore necessary to develop a dictionary between the two languages, in such a way to be able to address questions related to moduli dynamics within a given free-fermionic vacuum. While the equivalence is anticipated, writing a detailed dictionary is often non-trivial. A first attempt to establish such a link was done in [16–21] in the

context of $\mathcal{N} = 4$ toroidal compactification. In fact, in many quasi-realistic free-fermionic constructions the starting (four-dimensional) gauge symmetry is $\text{SO}(16) \times \text{SO}(16)$, rather than the more conventional $\text{E}_8 \times \text{E}_8$ symmetry — or, at times, the $\text{SO}(32)$ symmetry — typically considered in bosonic constructions. In the free-fermionic realisations the two choices correspond to different solutions of the modular invariance constraints. Then in terms of free bosons the two choices can be shown to depend on the possibility to turn on or off discrete torsion in certain freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds [19–21]. Alternatively, this amounts to different choices of Wilson lines and geometrical backgrounds.

Clearly, in order to build quasi-realistic chiral models, $\mathcal{N} = 4$ supersymmetry ought to be broken to $\mathcal{N} = 1$, and eventually to $\mathcal{N} = 0$. This can be achieved by performing a geometric $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection on the $\mathcal{N} = 4$ vacua. Additionally, this projection breaks the $\text{SO}(16) \times \text{SO}(16)$ gauge group to the more phenomenologically appealing $\text{SO}(10) \times \text{U}(1)^3 \times \text{SO}(16)$, while chiral matter emerges from the twisted sectors. Although in free-fermionic set-ups there are many consistent solutions with different low-energy chiral spectra [22–24], it seems that much fewer choices are present in the free bosonic case. However, this is in contrast with the expectation that the two formulations are equivalent. In particular, naively adding a \mathbb{Z}_2 geometric twist to the model of ref. [19–21] retains the vectorial representations in the massless spectrum rather than the spinorial ones [26].

In this paper we make a step forward in the direction of a better understanding of the connection between the formulation of the heterotic string in terms of free bosons and free fermions. A particular issue we would like to address is the recently proposed *spinor-vector duality* in heterotic-string vacua [27, 28], which was observed using the free fermionic language. This new duality relates vacua with spinorial and vectorial representations of orthogonal gauge groups, and it has been shown to hold in $\mathcal{N} = 2$ and $\mathcal{N} = 1$ free-fermionic models. It was also suggested that the spinor-vector duality can be thought of as being an extension of mirror symmetry [27, 28]. Indeed, mirror symmetry implies a change in the topology of the compactification manifold, that flips the sign of its Euler number. Equivalently, spinor-vector duality can be thought of as another topology-changing operation.

To date spinor-vector duality has not been studied in the orbifold language. In this paper we study this issue by analysing the $\text{E}_8 \times \text{E}_8$ heterotic string compactified on the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2$. The three \mathbb{Z}_2 operations correspond to the two supersymmetry preserving freely acting twists of ref. [19–21], while \mathbb{Z}''_2 reflects four internal coordinates and breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$. The E_8 , and $\text{SO}(16)$, symmetries are reduced by the \mathbb{Z}_2 twist to $\text{E}_7 \times \text{SU}(2)$, and $\text{SO}(12) \times \text{SO}(4)$, respectively. In this case the spinor and vector representations are both in the 56 representation of E_7 , that decomposes as $(32, 1) + (12, 2)$ under its maximal $\text{SO}(12) \times \text{SU}(2)$ subgroup. Here, the 32, and 12, are the spinorial, and vectorial, representations, of $\text{SO}(12)$, respectively. As the twisted sectors of the geometrical $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds preserve $\mathcal{N} = 2$ supersymmetry, it is sufficient to study the spinor-vector duality at this level rather than in the $\mathcal{N} = 1$ models, which are then obtained with an additional \mathbb{Z}_2 twist. The partition function associated to this $\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2$ has eight independent orbits, that admit seven discrete torsions taking the values ± 1 . Different choices of such discrete torsions clearly yield different spectra and, among those, there are some that retain the spinorial representation and others that retain vectorial one. As a

consequence, we note the existence of a transformation that maps between the cases, which reproduces the spinor-vector duality map observed in ref. [27, 28] within free fermionic construction. As in the free fermion case [27, 28], the spinor-vector duality exists at the $\mathcal{N} = 2$ level, which is obtained with a single \mathbb{Z}_2 twist acting on the internal coordinates. Actually, the heart of the spinor-vector splitting is in the choice of the $\mathcal{N} = 4$ vacuum, where $E_8 \times E_8$ is broken to $SO(16) \times SO(16)$. The additional \mathbb{Z}'_2 twist then selects either the spinorial or the vectorial representation of the resulting gauge group, and the spinor-vector duality map depends nontrivially on the discrete torsions, as we find in this paper. This is in a sense analogue with the mirror symmetry analysis of ref. [29], where the single discrete torsion of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ geometrical orbifold flips the Hodge numbers of the internal manifold.

Our paper is organised as follows: in section 2 we review the construction of “quasi-realistic” free-fermionic vacua and discuss the emergence of the recently discovered vector-spinor splitting as a freedom in the choice of generalised GSO (GGSO) phases. We then discuss, in section 3, equivalent constructions based on \mathbb{Z}_2^n orbifolds of free bosons and show explicitly how the vector-spinor splitting is, in this context, a consequence of the freedom of turning on or off different discrete torsions. Section 4 concludes with some comments, while in the appendix we list various combinations of characters that play a role in the constructions presented in section 3.

2 Spinor-vector splitting in free-fermionic models

In this section we discuss the spinor-vector splitting in free-fermionic models (see [11, 12] for a more detailed introduction). In the free-fermionic formulation of the heterotic string in four dimensions all the world-sheet degrees of freedom, required to cancel the conformal anomaly, are represented in terms of free fermions propagating on the string world sheet [1, 2]. In the light-cone gauge, the world-sheet degrees of freedom then consist of two transverse left-moving fermions $\psi_{1,2}^\mu$, superpartners of the space-time left-moving bosonic coordinates, together with additional 62 purely internal Majorana-Weyl fermions. Eighteen of them are left-moving,

$$\chi^{1,\dots,6}, \quad y^{1,\dots,6}, \quad \omega^{1,\dots,6},$$

while the remaining 44 are right-moving

$$\bar{y}^{1,\dots,6}, \quad \bar{\omega}^{1,\dots,6}, \quad \bar{\psi}^{1,\dots,5}, \quad \bar{\eta}^{1,2,3}, \quad \bar{\phi}^{1,\dots,8}.$$

Under parallel transport around a non-contractible loop on the toroidal world-sheet the fermionic fields pick up a phase, $f \rightarrow -e^{i\pi\alpha(f)} f$, $\alpha(f) \in (-1, +1]$. Each set of specified phases for all world-sheet fermions, around all the non-contractible loops is called the spin structure of the model. Such spin structures are usually given in the form of 64 dimensional boundary condition vectors, with each entry specifying the phase of the corresponding world-sheet fermion. The basis vectors are constrained by string consistency requirements, and completely determine the vacuum structure of the model. The physical spectrum is then obtained by applying suitable GGSO projections.

The boundary condition basis defining a typical “realistic free fermionic heterotic string model” is constructed in two stages. The first stage consists of the NAHE set, which is a set of five boundary condition basis vectors, $\{1, S, b_1, b_2, b_3\}$ [30, 31]

$$\begin{aligned} S &= \{\psi^{1,2}, \chi^{1,\dots,6}\}, \\ b_1 &= \{\psi^{1,2}, \chi^{1,2}, y^{3,\dots,6} | \bar{y}^{3,\dots,6}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\}, \\ b_2 &= \{\psi^{1,2}, \chi^{3,4}, y^{1,2}, \omega^{5,6} | \bar{y}^{1,2}, \bar{\omega}^{5,6}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\}, \\ b_3 &= \{\psi^{1,2}, \chi^{3,4}, \omega^{1,\dots,4} | \bar{\omega}^{1,\dots,4}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^3\}, \end{aligned}$$

where, for simplicity, only the fields with $\alpha(f) = 1$ are explicitly indicated, while those that are not listed have $\alpha(f) = 0$. After imposing the GSO projection, the gauge group is $SO(10) \times SO(6)^3 \times E_8$, and the vacuum enjoys $\mathcal{N} = 1$ supersymmetry. The second stage of the construction consists of adding to the NAHE set three (or four) additional boundary condition basis vectors, typically denoted by $\{\alpha, \beta, \gamma\}$. These additional basis vectors reduce the number of chiral generations to three, one from each of the sectors b_1, b_2 and b_3 , and simultaneously break the $SO(10)$ GUT symmetry to one of its subgroups [3–10].

The correspondence of the NAHE-based free fermionic models with the orbifold construction is illustrated by extending the NAHE set, $\{1, S, b_1, b_2, b_3\}$, by one additional boundary condition basis vector [16–18],

$$\xi_1 = \{\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}\}. \tag{2.1}$$

In this way, the orbifold construction involves an internal lattice with nontrivial background fields [32, 33]. Indeed, the subset of basis vectors

$$\{1, S, \xi_1, \xi_2\}, \quad \xi_2 = 1 + b_1 + b_2 + b_3 \tag{2.2}$$

generates a toroidally compactified model with $\mathcal{N} = 4$ space-time supersymmetry and $SO(12) \times E_8 \times E_8$ gauge group. Here the enhanced $U(1)^6 \rightarrow SO(12)$ gauge symmetry is precisely due to the choice of the internal $SO(12)$ lattice, with non trivial B_{ij} and G_{ij} backgrounds. Adding the two basis vectors b_1 and b_2 to the set (2.2) corresponds then to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model with standard embedding, and Hodge numbers $h_{11} = 27$ and $h_{21} = 3$. We note that the Euler characteristic of this orbifold differs from that of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at a generic point in the moduli space due to identification of fixed points by an internal lattice shift [16–21, 25].

The effect of the additional basis vector ξ_1 of eq. (2.1), is to separate the gauge degrees of freedom, spanned by the world-sheet fermions $\{\bar{\psi}^{1,\dots,5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \bar{\phi}^{1,\dots,8}\}$, from the internal compactified degrees of freedom $\{y, \omega | \bar{y}, \bar{\omega}\}^{1,\dots,6}$. In this construction, one actually has the freedom of flipping the sign of some GGSO phases, compatibly with modular invariance. In particular, the choice

$$c \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow -c \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{2.3}$$

breaks the $E_8 \times E_8$ gauge symmetry down to $SO(16) \times SO(16)$, that is instrumental for getting the GUT gauge group $SO(10)$ since, after the inclusion of the vectors b_1 and b_2 , $SO(16) \times SO(16) \rightarrow SO(10) \times U(1)^3 \times SO(16)$.

In the “realistic free fermionic models” this is achieved by the vector 2γ [16–18]

$$2\gamma = \{\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,4}\}, \tag{2.4}$$

that has the same effect of breaking $E_8 \times E_8$ gauge symmetry down to $SO(16) \times SO(16)$, and then to $SO(10) \times U(1)^3 \times SO(16)$ after the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ project is enforced.

The freedom in (2.3) actually corresponds to a discrete torsion. In fact, at the level of the $\mathcal{N}=4$ Narain model generated by the set (2.2), one can build two different vacua, \mathcal{Z}_+ and \mathcal{Z}_- , depending on the sign of the discrete torsion in eq. (2.3). The first, say \mathcal{Z}_+ , produces the $E_8 \times E_8$ model, whereas the second, say \mathcal{Z}_- , produces the $SO(16) \times SO(16)$ model. However, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist acts identically in the two models, and their physical characteristics differ only due to the discrete torsion eq. (2.3).

The projection induced by eqs. (2.4), or (2.3), has important phenomenological consequences in the free fermionic constructions that are relevant for orbifold models. In the case of \mathcal{Z}_+ , the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold breaks the observable E_8 symmetry to $E_6 \times U(1)^2$. The chiral matter states are contained in the 27 representation of E_6 , which decomposes as

$$27 = 16_{\frac{1}{2}} + 10_{-1} + 1_2 \tag{2.5}$$

under its $SO(10) \times U(1)$ subgroup, where the spinorial 16 and vectorial 10 representations of $SO(10)$ contain the Standard Model fermion and Higgs states, respectively. The projection induced by (2.3) in \mathcal{Z}_- entails that either the spinorial or the vectorial representation survives the GSO projection at a given fixed point. Hence, this projection operates a Higgs-matter splitting mechanism [34] in the phenomenological free fermionic models.

Semi-realistic four-dimensional heterotic vacua have also been built using orbifold technique, based on a choice of gauge bundle and geometrical twist. These constructions are based essentially on the $E_8 \times E_8$ heterotic string, and the breaking of E_8 is achieved by suitable Wilson lines [13, 14] (for constructions based on the $SO(32)$ heterotic string see e.g. [35–37]). In this set-ups, different heterotic vacua can be connected by choices of different gauge bundles and Wilson lines. Although the equivalence of geometrical \mathbb{Z}_2 's orbifold constructions and free-fermionic constructions is rather obvious, a explicit link between the two approaches is still missing and, in particular, to date it is not know how to interpret the spinor-vector duality in the realm of orbifold compactification. In the next section, we shall try to fill this gap by analysing a specific \mathbb{Z}_2^n orbifold and will identify the spinor-vector splitting in terms of discrete torsion. Connecting the choices of discrete torsion to the choice of gauge bundles, along the lines of [15], is an interesting open problem that we are able to answer only in the simple case of $\mathcal{N}=4$ vacua.

3 Spinor-vector duality in four-dimensional $\mathcal{N}=2$ orbifold vacua

As anticipated, the emergence of spinorial representations in the twisted sectors of four-dimensional $\mathcal{N}=1$ heterotic vacua based on \mathbb{Z}_2^n orbifolds has its origin in the simpler context of vacua with eight supercharges, where the E_8 gauge group is directly broken to an orthogonal one.

To be specific, we consider the $E_8 \times E_8$ heterotic string compactified on the $(T^4 \times T^2)/\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2$ orbifold. The factorisation of the internal T^6 in terms of the product of a four-torus times a two-torus is suggested by the way the three \mathbb{Z}_2 's act on the various degrees of freedom. In particular, the free action generated by $\mathbb{Z}_2 \times \mathbb{Z}'_2$, with

$$\mathbb{Z}_2 \ni g = (-1)^{F_1} \delta, \quad \mathbb{Z}'_2 \ni g' = (-1)^{F_2} \delta,$$

where $F_{1,2}$ flips the sign of the spinorial representation in $E_8 = \text{Spin}(16)/\mathbb{Z}_2$ and δ shifts the compact x^4 coordinate by half of its period, spontaneously break the $E_8 \times E_8$ gauge group into $\text{SO}(16) \times \text{SO}(16)$, while preserving the original $\mathcal{N} = 4$ supersymmetries in four-dimensions. The additional \mathbb{Z}''_2 factor, instead, twists also the space-time degrees of freedom and preserves only $\mathcal{N} = 2$ supersymmetries. Its generator g'' reverts the sign of the four internal coordinates x^i , $i = 6, 7, 8, 9$, and, at the same time, breaks one $\text{SO}(16)$ gauge factor (the first one, say) into $\text{SO}(12) \times \text{SO}(4)$.

To implement the action of the $\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2$ orbifold, it is convenient to break the ten-dimensional $\text{SO}(8)$ little group into $\text{SO}(4) \times \text{SO}(4)$, where the second $\text{SO}(4)$ factor reflects the symmetry of the internal T^4 , while the first $\text{SO}(4)$ factor corresponds to the ‘‘enhanced’’ little group of $\mathcal{M}_{1,3} \times T^2$. At the same time, the first $\text{Spin}(16)$ group factor is broken into $\text{Spin}(12) \times \text{Spin}(4)$. As a result, the one-loop partition function can be written in terms of the familiar space-time characters, Q_o, Q_v, Q_s and Q_c , and the gauge-group ones, $\chi_i^o, \chi_i^v, \chi_i^s, \chi_i^c$ and $\xi_{1,g'}^{o,v}$. For completeness, their explicit expression in terms of $\text{SO}(2n)$ characters [38, 39] is given in the appendix. The corresponding genus one partition function thus reads

$$\mathcal{Z} = \frac{1}{8} \sum_{\alpha} \mathcal{Z}_{\alpha},$$

where α labels the eight (un)twisted sectors and each amplitude \mathcal{Z}_{α} is given explicitly by

$$\begin{aligned} \mathcal{Z}_1 = & \left\{ (\bar{Q}_o + \bar{Q}_v) [\chi_1^o \xi_1^o + \chi_g^o \xi_{g'}^o + (-1)^m (\chi_g^o \xi_1^o + \chi_1^o \xi_{g'}^o)] \Lambda^{(4,4)} \right. \\ & \left. + (\bar{Q}_o - \bar{Q}_v) [\chi_{g''}^o \xi_1^o + \chi_{g g''}^o \xi_{g'}^o + (-1)^m (\chi_{g g''}^o \xi_1^o + \chi_{g''}^o \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_2} \right|^4 \right\} \\ & \times \Lambda^{(2,2)}, \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_g = & \left\{ (\bar{Q}_o + \bar{Q}_v) [\chi_1^v \xi_1^o - \epsilon_1 \chi_g^v \xi_{g'}^o - (-1)^m (\chi_g^v \xi_1^o - \epsilon_1 \chi_1^v \xi_{g'}^o)] \Lambda^{(4,4)} \right. \\ & \left. + (\bar{Q}_o - \bar{Q}_v) [\epsilon_2 \chi_{g''}^v \xi_1^o - \epsilon_3 \chi_{g g''}^v \xi_{g'}^o - (-1)^m (\epsilon_2 \chi_{g g''}^v \xi_1^o - \epsilon_3 \chi_{g''}^v \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_2} \right|^4 \right\} \\ & \times \Lambda_{1/2}^{(2,2)}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathcal{Z}_{g'} = & \left\{ (\bar{Q}_o + \bar{Q}_v) [\chi_1^o \xi_1^v - \epsilon_1 \chi_g^o \xi_{g'}^v + (-1)^m (\epsilon_1 \chi_g^o \xi_1^v - \chi_1^o \xi_{g'}^v)] \Lambda^{(4,4)} \right. \\ & \left. + (\bar{Q}_o - \bar{Q}_v) [\epsilon_4 \chi_{g''}^o \xi_1^v - \epsilon_5 \chi_{g g''}^o \xi_{g'}^v + (-1)^m (\epsilon_5 \chi_{g g''}^o \xi_1^v - \epsilon_4 \chi_{g''}^o \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_2} \right|^4 \right\} \end{aligned} \tag{3.2}$$

$$\begin{aligned}
 & \times \Lambda_{1/2}^{(2,2)}, \\
 \mathcal{Z}_{g g'} = & \left\{ (\bar{Q}_o + \bar{Q}_v) [\chi_1^v \xi_1^v + \chi_g^v \xi_{g'}^v - \epsilon_1 (-1)^m (\chi_g^v \xi_1^v + \chi_1^v \xi_{g'}^v)] \Lambda^{(4,4)} \right. \\
 & \left. + (\bar{Q}_o - \bar{Q}_v) [\epsilon_6 (\chi_{g''}^v \xi_1^v + \chi_{g g''}^v \xi_{g'}^v) + \epsilon_7 (-1)^m (\chi_{g g''}^v \xi_1^v + \chi_{g''}^v \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_2} \right|^4 \right\} \\
 & \times \Lambda^{(2,2)}, \\
 \mathcal{Z}_{g''} = & \left\{ (\bar{Q}_s + \bar{Q}_c) [\chi_1^c \xi_1^o - \epsilon_6 \chi_g^c \xi_{g'}^o - (-1)^m (\epsilon_2 \chi_g^c \xi_1^o - \epsilon_4 \chi_1^c \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_4} \right|^4 \right. \\
 & \left. + (\bar{Q}_s - \bar{Q}_c) [\chi_{g''}^c \xi_1^o - \epsilon_6 \chi_{g g''}^c \xi_{g'}^o - (-1)^m (\epsilon_2 \chi_{g g''}^c \xi_1^o - \epsilon_4 \chi_{g''}^c \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_3} \right|^4 \right\} \\
 & \times \Lambda^{(2,2)}, \\
 \mathcal{Z}_{g g''} = & \left\{ (\bar{Q}_s + \bar{Q}_c) [\chi_1^s \xi_1^o - \epsilon_7 \chi_g^s \xi_{g'}^o + (-1)^m (\epsilon_2 \chi_g^s \xi_1^o + \epsilon_5 \chi_1^s \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_4} \right|^4 \right. \\
 & \left. + (\bar{Q}_s - \bar{Q}_c) [\epsilon_2 \chi_{g''}^s \xi_1^o + \epsilon_5 \chi_{g g''}^s \xi_{g'}^o + (-1)^m (\chi_{g g''}^s \xi_1^o - \epsilon_7 \chi_{g''}^s \xi_{g'}^o)] \left| \frac{2\eta}{\vartheta_3} \right|^4 \right\} \\
 & \times \Lambda_{1/2}^{(2,2)}, \\
 \mathcal{Z}_{g' g''} = & \left\{ (\bar{Q}_s + \bar{Q}_c) [\chi_1^c \xi_1^v - \epsilon_7 \chi_g^c \xi_{g'}^v - (-1)^m (\epsilon_3 \chi_g^c \xi_1^v + \epsilon_4 \chi_1^c \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_4} \right|^4 \right. \\
 & \left. + (\bar{Q}_s - \bar{Q}_c) [\epsilon_4 \chi_{g''}^c \xi_1^v + \epsilon_3 \chi_{g g''}^c \xi_{g'}^v + (-1)^m (\epsilon_7 \chi_{g g''}^c \xi_1^v - \chi_{g''}^c \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_3} \right|^4 \right\} \\
 & \times \Lambda_{1/2}^{(2,2)},
 \end{aligned} \tag{3.3}$$

and, finally,

$$\begin{aligned}
 \mathcal{Z}_{g' g' g''} = & \left\{ (\bar{Q}_s + \bar{Q}_c) [\chi_1^s \xi_1^v - \epsilon_6 \chi_g^s \xi_{g'}^v + (-1)^m (\epsilon_3 \chi_g^s \xi_1^v - \epsilon_5 \chi_1^s \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_4} \right|^4 \right. \\
 & \left. + (\bar{Q}_s - \bar{Q}_c) [\epsilon_6 \chi_{g''}^s \xi_1^v - \chi_{g g''}^s \xi_{g'}^v + (-1)^m (\epsilon_5 \chi_{g g''}^s \xi_1^v - \epsilon_3 \chi_{g''}^s \xi_{g'}^v)] \left| \frac{2\eta}{\vartheta_3} \right|^4 \right\} \\
 & \times \Lambda^{(2,2)}.
 \end{aligned}$$

Before we analyse the properties of the spectrum of this heterotic orbifold, it is convenient to explain the notation and the origin of the ϵ 's signs. For convenience, let us take the amplitude \mathcal{Z}_g . It is actually a short-hand notation for

$$\mathcal{Z}_g = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^4} \sum_{m_4, m_5, n_4, n_5} \left[\frac{(\bar{Q}_o + \bar{Q}_v)}{\bar{\eta}^2} \frac{(\chi_1^v - (-1)^{m_4} \chi_g^v) (\xi_1^o + \epsilon_1 (-1)^{m_4} \xi_{g'}^o)}{\eta^2} \right] \Lambda^{(4,4)}$$

$$\begin{aligned}
 & + \frac{(\bar{Q}_o - \bar{Q}_v)}{\bar{\eta}^2} \frac{(\chi_{g''}^v - (-1)^{m_4} \chi_{g g''}^v)}{\eta^2} \frac{(\epsilon_2 \xi_1^o + \epsilon_3 (-1)^{m_4} \xi_{g'}^o)}{\eta^2} \left| \frac{2\eta}{\vartheta_2} \right|^4 \Big] \\
 & \times \Lambda_{m_4, m_5; n_4 + \frac{1}{2}, n_5}^{(2,2)},
 \end{aligned}$$

where the eta functions in the denominators count the contribution of the non-compact world-sheet bosons in the light-cone gauge, while

$$\Lambda^{(4,4)} = \sum_{m_i, n^i} \frac{q^{\frac{\alpha'}{4} p_L^2} \bar{q}^{\frac{\alpha'}{4} p_R^2}}{\eta^4 \bar{\eta}^4}$$

denotes the four-dimensional Narain lattice associated to the directions x^i , $i = 6, 7, 8, 9$, upon which g'' has a non-trivial action. Finally, the (shifted) zero modes associated to the two remaining compact coordinates fill the lattice

$$\Lambda_{m_4, m_5; n_4 + b, n_5}^{(2,2)} = \frac{q^{\frac{\alpha'}{4} \left(\frac{m_4}{R_4} + \frac{(n_4 + b) R_4}{\alpha'} \right)^2} \bar{q}^{\frac{\alpha'}{4} \left(\frac{m_4}{R_4} - \frac{(n_4 + b) R_4}{\alpha'} \right)^2}}{\eta \bar{\eta}} \frac{q^{\frac{\alpha'}{4} \left(\frac{m_5}{R_5} + \frac{n_5 R_5}{\alpha'} \right)^2} \bar{q}^{\frac{\alpha'}{4} \left(\frac{m_5}{R_5} - \frac{n_5 R_5}{\alpha'} \right)^2}}{\eta \bar{\eta}},$$

where $b = 0$ in the untwisted, g'' , $g g'$ and $g g' g''$ twisted sectors, while $b = \frac{1}{2}$ in the g , g' , $g g''$ and $g' g''$ twisted sectors.

The signs ϵ_i reflect the possibility of turning on discrete torsion in this \mathbb{Z}_2^3 orbifold. Clearly, they affect the massless and massive spectrum and in particular the gauge-group representations of the twisted matter.

This can be neatly seen by writing a q -series expansion of the various contributions to the partition function, and keeping for simplicity only the low-lying states. In particular, noting that

$$\begin{aligned}
 q^{n/12} V_{2n} & \sim 2n q^{1/2} + O(q^{3/2}), & q^{n/12} O_{2n} & \sim q^{-1} + n(2n-1) + O(q), \\
 q^{n/12} S_{2n} & \sim 2^{n-1} q^{n/2} + O(q^{n/2+1}), & q^{n/12} C_{2n} & \sim 2^{n-1} q^{n/2} + O(q^{n/2+1}),
 \end{aligned}$$

and using similar expansions for the theta and eta functions, one finds that only the untwisted, $g g'$, g'' and $g g' g''$ twisted sectors actually yield massless states. More in details, the leading contributions to the amplitudes read

$$\mathcal{L}_{(0)} = \mathcal{L}_{(0)1} + \mathcal{L}_{(0)g g'} + \mathcal{L}_{(0)g''} + \mathcal{L}_{(0)g g' g''},$$

where

$$\begin{aligned}
 \mathcal{L}_{(0)1} & \sim \bar{Q}_o O_4 O_{12} O_{16} + \bar{Q}_v V_4 V_{12} O_{16} + \text{massive}, \\
 \mathcal{L}_{(0)g g'} & \sim \bar{Q}_o \left[O_4 V_{12} V_{16} \frac{1 - \epsilon_1 + \epsilon_6 + \epsilon_7}{4} + V_4 O_{12} V_{16} \frac{1 - \epsilon_1 - \epsilon_6 - \epsilon_7}{4} \right] \\
 & + \bar{Q}_v \left[O_4 V_{12} V_{16} \frac{1 - \epsilon_1 - \epsilon_6 - \epsilon_7}{4} + V_4 O_{12} V_{16} \frac{1 - \epsilon_1 + \epsilon_6 + \epsilon_7}{4} \right] + \text{massive}, \\
 \mathcal{L}_{(0)g''} & \sim 16 \bar{Q}_s \left[O_4 S_{12} O_{16} \frac{1 - \epsilon_2 + \epsilon_4 - \epsilon_6}{4} + C_4 V_{12} O_{16} \frac{1 + \epsilon_2 + \epsilon_4 + \epsilon_6}{4} \right] \\
 & + 16 \bar{Q}_s \left[S_4 O_{12} O_{16} \frac{1 + \epsilon_2 + \epsilon_4 + \epsilon_6}{4} \right] + \text{massive},
 \end{aligned}$$

and, finally,

$$\mathcal{Z}_{(0)gg'g''} \sim 16 \bar{Q}_s C_4 O_{12} V_{16} \frac{1 - \epsilon_3 - \epsilon_5 + \epsilon_6}{4} + \text{massive}.$$

The untwisted sector, independent of the discrete torsion, comprises an $\mathcal{N}=2$ supergravity multiplet, coupled to vector multiplets in the adjoint representation of the gauge group $G = U(1)^2 \times SO(4) \times SO(12) \times SO(16)$, and hypermultiplets in the representation $(4, 12, 1)$.

The twisted matter includes neutral hypermultiplets associated to the deformations of $K3$, together with hypermultiplets charged with respect to G , whose representations depend on the choice of the discrete torsions. Clearly, for the partition function to be real the ϵ 's can only be signs, while demanding that \mathcal{Z} have a physical interpretation in terms of a proper counting of states, the various coefficients of the q^α terms must be integers, positive for bosons and negative for fermions. This clearly implies that the combinations of discrete torsion, like those appearing in $\mathcal{Z}_{(0)}$ should equal 0 or 1. Finally, the last requirement we want to impose on the ϵ 's is that the gauge group be the smallest one. In fact, if any of the combinations in the first line of $\mathcal{Z}_{(0),gg'}$ is different than zero, the gauge group is enhanced to $SO(16) \times SO(16)$ or to $SO(12) \times SO(20)$. As we shall see momentarily, this possibility is already present at the level of $\mathcal{N}=4$ vacua, and corresponds to different discrete values of Wilson lines. Taking all these constraints into account, the possible choices of discrete torsion turn out to be

$$\epsilon_1 = 1, \quad \epsilon_7 = -\epsilon_6, \quad \epsilon_4 = \epsilon_5,$$

and

$$\begin{aligned} sol_1 &= (-1, -1, +1, -1), & sol_2 &= (+1, +1, -1, -1), \\ sol_3 &= (+1, +1, +1, +1), & sol_4 &= (-1, -1, -1, +1), \end{aligned}$$

where

$$sol_i = (\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6).$$

As a result, the massless twisted spectra depend on the allowed combination sol_i of signs, and are listed in table 1. This is a neat instance of spinor-vector duality and is at the heart of Higgs-matter splitting in more realistic vacua. Let us note that for sol_3 extra 8×8 neutral massless hypermultiplets appear from the twisted sector

$$\bar{Q}_s S_4 O_{12} O_{16},$$

hence keeping a total number of massless degrees of freedom unchanged. From table 1 it is observed that under the different possibilities of the discrete torsions the number of massless degrees of freedom is preserved, except sol_2 that does not have any twisted massless states, similar to what is observed in the free fermionic classification of [27, 28]. Let us note that there are no other massless twisted neutral hypermultiplets in any of sol_i .

To further break supersymmetry, and get more realistic chiral models, it is enough to act with an additional \mathbb{Z}_2''' that twists the coordinates $x^{4,5,6,7}$, say, and breaks the $SO(12)$ gauge group to the more phenomenological $SO(10)$, while leaving untouched the hidden

solution	reps of massless charged hypermultiplets
sol_1	$8 \times (1, 32, 1)$
sol_2	—
sol_3	$8 \times [(2, 12, 1) + 4 \times (2, 1, 1)]$
sol_4	$8 \times (2, 1, 16)$

Table 1. Charged massless twisted spectrum for the T^6/\mathbb{Z}_2^3 heterotic orbifold, for different choices of discrete torsion. The non-Abelian gauge group is $G = \text{SO}(4) \times \text{SO}(12) \times \text{SO}(16)$ and the vacuum configurations also include universal (untwisted) charged hypermultiplets in the representation $(4, 12, 1)$.

sector. Although, additional discrete torsions can be turned on for this \mathbb{Z}_2^4 model, in the simplest instance where one considers only the seven signs previously introduced, the resulting massless chiral spectrum for the choice sol_1 includes an $\text{SO}(10)$ spinorial representation since, under the action of \mathbb{Z}_2''' , $32 \rightarrow 16 + \overline{16}$, and only one spinorial eventually survives the overall orbifold projection in a chiral model. On the other hand, the solutions sol_3 and sol_4 would only include matter in vectorial (Higgs-like) representation.

Furthermore, the reduction of the number of chiral families, or alternatively the change of the topology of the Calabi-Yau manifold, can be achieved, as usual, through the implementation of additional shift symmetries. They do not twist any internal coordinate and the only effect on the spectrum consists in reducing the number of families of twisted chiral matter through an identification of fixed points. In terms of free fermionic constructions, this is equivalent to the inclusion of the $\{\alpha, \beta, \gamma\}$ system of boundary condition basis vectors to the NAHE set, as discussed in the previous section.

Before we conclude, let us make a brief remark on the interpretation of the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ freely-acting orbifold of the $E_8 \times E_8$ heterotic string. As already stated several times, this orbifold projection does not break any of the original supersymmetries, and therefore corresponds to a nine-dimensional vacuum with 16 supercharges. Depending on the value of the discrete torsion¹ one gets the models

$$\mathcal{L}_{\epsilon_1=+1} = (\bar{V}_8 - \bar{S}_8) [O_{32} A_{2m,n} + S_{32} A_{2m+1,n} + V_{32} A_{2m+1,n+1/2} + C_{32} A_{2m,n+1/2}] ,$$

with an $\text{SO}(32)$ gauge group,

$$\begin{aligned} \mathcal{L}_{\epsilon_1=-1} = (\bar{V}_8 - \bar{S}_8) [& (O_{16} O_{16} + C_{16} C_{16}) A_{2m,n} + (S_{16} S_{16} + V_{16} V_{16}) A_{2m+1,n} \\ & + (C_{16} O_{16} + O_{16} C_{16}) A_{2m,n+1/2} + (V_{16} S_{16} + S_{16} V_{16}) A_{2m+1,n+1/2}] , \end{aligned}$$

with a broken $\text{SO}(16) \times \text{SO}(16)$ gauge group.

However, $\mathcal{N} = 4$ vacua are characterised by a moduli space, uniquely fixed by its dimension, and therefore by the dimension of the compactification torus and by the rank of the gauge group. Indeed, the heterotic vacua obtained as an S^1 and $S^1/\mathbb{Z}_2 \times \mathbb{Z}'_2$ compactification, with or without discrete torsion, are all continuously connected. In this respect,

¹The discrete torsion present in this $\mathbb{Z}_2 \times \mathbb{Z}'_2$ actually corresponds to the sign ϵ_1 in eqs. (3.1), (3.2) and (3.3).

the ϵ_1 discrete torsion has a natural geometrical description in terms of discrete values of otherwise continuous Wilson lines along the compact S^1 . It is tempting to interpret also the remaining signs as specific choices of gauge bundles and/or Wilson lines as in [15]. Although this connection seems quite natural, it is less evident than in the $\mathcal{N}=4$ case and requires further analysis.

4 Conclusions

Heterotic string theory is unique among the perturbative string constructions since it gives rise naturally to the GUT embedding of the Standard Model matter states in $SO(10)$ and E_6 representations in a perturbative, and thus calculable, set-up. Grand unification is well supported by the pattern of observed fermion and gauge boson charges. In the framework of $SO(10)$ gauge theory all the matter states of a single generation are unified in the 16 spinorial representation and, a priori, one needs only two types of representations, the spinorial 16 and the vectorial 10 representations, to embed the Standard Model matter and Higgs spectrum. The framework of E_6 grand unification has the further property of incorporating the 16 matter and 10 Higgs states into the 27 representation of E_6 .

As the observed gauge symmetry at low energies consists solely of the Standard Model one, its embedding into a grand unification group necessitates that the larger GUT symmetry be broken. Moreover, grand unification introduces additional difficulties with proton decay and neutrino masses. The GUT symmetry breaking and the miscellaneous issues typically require the introduction of large representations, like the 126 of $SO(10)$, or the 351 of E_6 .

By producing the gauge and matter structures that arise in Grand Unified Theories, heterotic string theories offer new possibilities to tame the problems that arise in field theory GUTs. To this end, to understand the various alternatives offered by string theory, it is important to construct quasi-realistic string models and investigate their properties in detail. The main approaches to this program are free-fermionic [1, 2] and bosonic [32, 33, 40, 41] constructions, as well as interacting [42–45] world-sheet conformal field theories. The heterotic-string models in their free fermionic formulation [3–10], first constructed over two decades ago, are among the most realistic string vacua constructed to date, though, in recent years comparable quasi-realistic models have also been constructed using free world-sheet bosons [13, 14, 35–37]. It should be stressed, however, that the two formulations are closely related and that the corresponding string vacua can be described equivalently using both approaches. Therefore it is natural to assume that for every string model constructed using free world-sheet fermions an identical vacuum exists constructed using free world-sheet bosons. Indeed, the free fermionic models correspond to \mathbb{Z}_2^n toroidal orbifolds at special points in the moduli space.

While the free-fermionic approach can be straightforwardly implemented as an algebraic set of conditions that facilitate the scan of phenomenological properties, the free boson approach is more readily adaptable to explore the underlying moduli dynamics. It is therefore compelling to develop a dictionary between the two languages. Although the equivalence is anticipated, writing a detailed dictionary is often non-trivial. In this paper we investigated this aspect in some detail. An important feature in the quasi-realistic free

fermionic models is the breaking of the $E_8 \times E_8$ symmetry to $SO(16) \times SO(16)$ at the level of the underlying $\mathcal{N}=4$ toroidal compactification. This breaking is realised in the bosonic construction in terms of freely acting orbifolds, or alternatively in terms of Wilson lines. Matter states arise in the free fermion models from \mathbb{Z}_2 twisted sectors, which break the $\mathcal{N}=4$ space-time supersymmetry to $\mathcal{N}=2$. The next step in building the dictionary between the two classes of models is therefore to add a \mathbb{Z}_2 orbifold to the two freely acting orbifolds of [19–21]. However, it turns out that the construction is not straightforward. In the free fermionic models the one-loop partition functions are generated in terms of boundary condition basis vectors and GGSO phases. One then builds a space of modular invariant partition functions, with differing physical characteristics. In the bosonic representation, on the other hand, a variety of vacua arise from the freedom to choose the background fields and from the existence of disconnected modular orbits. The detailed correspondence between the two representations, while formally well understood and established, is nevertheless non-trivial and often obscure. In this paper we addressed this issue with respect to the twisted matter states and spinor-vector duality, first observed in the classification of free fermionic models [22–24, 27, 28]. The spinor-vector duality is a property over the full space of vacua generated by the given set of basis vectors [27, 28] and corresponds to maps between different choices of GGSO projection coefficients. In the orbifold language, as demonstrated here, it corresponds to different choices of discrete torsions, thus extending the map of [29]. Two issues are of interest here. The first is to improve the understanding of the detailed correspondence between the free fermion GGSO projection coefficients and the orbifold discrete torsions. The second is to understand the spinor-vector duality in geometrical terms. We anticipate that this should entail an action on the internal moduli plus an action on the bundle that generates the heterotic-string gauge degrees of freedom. We note that the existence of the spinor-vector duality raises basic questions in regard to the relation of the string vacua to the low energy effective field theory. While in the effective field theory the two models identified under the spinor-vector duality map are clearly distinct, from the string point of view they are closely related. This is exemplified by the fact that the number of degrees of freedom is preserved under the map. Thus, whereas the spinor of $SO(12)$ contains 32 states and the vector contains only $2 \times 12 = 24$, the vector representation is augmented by additional 8 $SO(12)$ singlets, that correct the mismatch. The issue can be further explored by studying the effect of the duality map on interactions. Another issue of further interest is the relation of the spinor-vector duality to triality of $SO(8)$. This question was briefly explored in the context of the free fermionic classification [27, 28] by breaking the untwisted gauge degrees of freedom to four $SO(8)$ factors. We hope to address these issues in future publications.

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A List of characters

In this appendix, we list the space-time and gauge-group characters that enter in the T^6/\mathbb{Z}_2^3 partition function. For the definition of the $SO(2n)$ characters in terms of eta and theta functions, as well as for their modular properties, we report the interested reader to [38, 39].

Space-time (anti-holomorphic) characters

$$\bar{Q}_o = \bar{V}_4 \bar{O}_4 - \bar{C}_4 \bar{C}_4, \quad \bar{Q}_s = \bar{O}_4 \bar{C}_4 - \bar{S}_4 \bar{O}_4, \quad (\text{A.1})$$

$$\bar{Q}_v = \bar{O}_4 \bar{V}_4 - \bar{S}_4 \bar{S}_4, \quad \bar{Q}_c = \bar{V}_4 \bar{S}_4 - \bar{C}_4 \bar{V}_4. \quad (\text{A.2})$$

Gauge-group (holomorphic) characters associated to the first $E_8 \rightarrow SO(4) \times SO(12)$ factor

$$\begin{aligned} \chi_1^o &= O_4 O_{12} + V_4 V_{12} + S_4 S_{12} + C_4 C_{12}, \\ \chi_g^o &= O_4 O_{12} + V_4 V_{12} - S_4 S_{12} - C_4 C_{12}, \\ \chi_{g''}^o &= O_4 O_{12} - V_4 V_{12} - S_4 S_{12} + C_4 C_{12}, \\ \chi_{g'g''}^o &= O_4 O_{12} - V_4 V_{12} + S_4 S_{12} - C_4 C_{12}, \\ \chi_1^v &= O_4 V_{12} + V_4 O_{12} + S_4 C_{12} + C_4 S_{12}, \\ \chi_g^v &= O_4 V_{12} + V_4 O_{12} - S_4 C_{12} - C_4 S_{12}, \\ \chi_{g''}^v &= O_4 V_{12} - V_4 O_{12} - S_4 C_{12} + C_4 S_{12}, \\ \chi_{g'g''}^v &= O_4 V_{12} - V_4 O_{12} + S_4 C_{12} - C_4 S_{12}, \\ \chi_1^c &= O_4 S_{12} + V_4 C_{12} + S_4 O_{12} + C_4 V_{12}, \\ \chi_g^c &= O_4 S_{12} + V_4 C_{12} - S_4 O_{12} - C_4 V_{12}, \\ \chi_{g''}^c &= O_4 S_{12} - V_4 C_{12} - S_4 O_{12} + C_4 V_{12}, \\ \chi_{g'g''}^c &= O_4 S_{12} - V_4 C_{12} + S_4 O_{12} - C_4 V_{12}, \end{aligned}$$

and

$$\begin{aligned} \chi_1^s &= O_4 C_{12} + V_4 S_{12} + S_4 V_{12} + C_4 O_{12}, \\ \chi_g^s &= O_4 C_{12} + V_4 S_{12} - S_4 V_{12} - C_4 O_{12}, \\ \chi_{g''}^s &= O_4 C_{12} - V_4 S_{12} - S_4 V_{12} + C_4 O_{12}, \\ \chi_{g'g''}^s &= O_4 C_{12} - V_4 S_{12} + S_4 V_{12} - C_4 O_{12}. \end{aligned}$$

Gauge-group (holomorphic) characters associated to the second $E_8 \rightarrow SO(16)$ factor

$$\begin{aligned} \xi_1^o &= O_{16} + S_{16}, & \xi_{g'}^o &= O_{16} - S_{16}, \\ \xi_1^v &= V_{16} + C_{16}, & \xi_{g'}^v &= V_{16} - C_{16}. \end{aligned}$$

Notice that the characters $\xi_{1,g'}^{o,v}$ are exactly equal to $\chi_{1,g}^{o,v}$. We use a different notation only to stress that on the second E_8 only the g' generator acts non-trivially and therefore the group is simply broken to $SO(16)$.

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