Properties of Lie Algebras of Vector Fields from Lie Determining System

by

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Abstract

Differential equations (DEs) are involved in many mathematical applications to the real world such as fluid flow, stock pricing, etc. However, solving DEs explicitly can be extremely difficult. One common approach to this problem is to analyse solutions of DEs using their symmetries. Infinitesimal symmetries of DEs (or other objects) obey a Lie 'determining system' of linear homogeneous partial DEs (LHPDEs). The solutions of such systems, solution vector fields, form a Lie algebra $\mathcal{L}$. Explicit integration of the determining system is often achieved, however integration is not a finite algorithmic process and so it is not suited to computer algebra. It is therefore of great interest in computer algebra to systematically extract information about solution vector fields (a.k.a. Lie algebras of vector fields LAVFs) directly from the determining system L without solving it. Our aim is to provide such a general toolkit as a computer algebra package.

In this thesis, we exhibit several algorithms that work directly at the level of the determining system L. The key idea is to use the Existence and Uniqueness theorem for 'differentially complete' LHPDEs $\widehat{L}$ derived from L so the properties of a LAVF can be explored without solving them. A Maple command `risimp` is used for computing $\widehat{L}$, although others could also be used. A procedure is given that once can decide if a system specifies a Lie algebra $\mathcal{L}$, if $\mathcal{L}$ is abelian and if a system L' specifies an ideal in $\mathcal{L}$. Algorithms are described that compute determining systems for transporter, Lie product and Killing orthogonal subspace associated with $\mathcal{L}$. This gives a systematic calculus for Lie determining systems, enabling computation of the determining systems for normalisers, centralisers, centre, derived algebra, solvable radical and key series (derived series, lower/upper central series) for $\mathcal{L}$. Our methods thereby give algorithmic access to new geometrical invariants of the symmetry action. We also discuss and generalise Reid's method for finding structure constants at the determining system level. We then revise our algorithms so they accept a partially solved system.

The second part of the thesis involves the development of a computer algebra toolkit as Maple packages for these algorithms. Our main package works entirely on the determining system, it consists of more than 60 methods for exploring LAVFs, and approximately 5000 lines of code. Two other generic packages were also developed for servicing the main package. All our packages are in object-oriented Maple and are designed according to modern software engineering principles. A proper test suite is developed for these packages, and a complete set of Maple help pages are produced as a Maple help database for each package. The package's design aids its robustness, user-friendly, reliability and also facilitates its extension to other applications.

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