Spacetime Initial Data and Quasispherical Coordinates.

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by

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Preface.

In General Relativity, the Einstein field equations allow us to study the evolution of a spacelike 3-manifold, provided that its metric and extrinsic curvature satisfy a system of geometric constraint equations. The so-called Einstein constraint equations, arise as a consequence of the fact that the 3-manifold in question is necessarily a submanifold of the spacetime its evolution defines.

This thesis is devoted to a study of the structure of the Einstein constraint system in the special case when the spacelike 3-manifold also satisfies the quasispherical ansatz of Bartnik [B93]. We make no mention of the generality of this gauge; the extent to which the quasispherical ansatz applies remains an open problem.

After imposing the quasispherical gauge, we give an argument to show that the resulting Einstein constraint system may be viewed as a coupled system of partial differential equations for the parameters describing the metric and second fundamental form. The hencenamed quasisperical Einstein constraint system, consists of a parabolic equation, a first order elliptic system and (essentially) a system of ordinary differential equations.

The question of existence of solutions to this system naturally arises and we provide a partial answer to this question. We give conditions on the initial data and prescribable fields under which we may conclude that the quasispherical Einstein constraint system is uniquley solvable, at least in a region surrounding the unit sphere.

The proof of this fact is centred on a linear iterative system of partial differential equations, which also consist of a parabolic equation, a first order elliptic system and a system of ordinary differential equations. We prove that this linear system consistently defines a sequence, and show via a contraction mapping argument, that this sequence must converge to a fixed point of the iteration. The iteration, however, has been specifically designed so that any fixed point of the iteration coincides with a solution of the quasispherical Einstein constraints.

The contraction mapping argument mentioned above, relies heavily on a priori estimates for the solutions of linear parabolic equations. We generalise and extend known results concerning parabolic equations to establish special a priori estimates which relate a useful property: the L^2-Sobolev regularity of the solution of a parabolic equation is greater than that of the coefficients of the elliptic operator, provided that the initial data is sufficiently regular. This 'smoothing' property of linear parabolic equations along with several estimates from elliptic and ordinary differential equation theory form the crucial ingredients needed in the proof of the existence of a fixed point of the iteration.

We begin in chapter one by giving a brief review of the extensive literature concerning the initial value problem in General Relativity. We go on, after mentioning two of the traditional methods for constructing spacetime initial data, to introduce the notion of a quasispherical foliation of a 3-manifold and present the Einstein constraint system written in terms of this gauge.

In chapter two we introduce the various inequalities and tracts of analysis we will make use of in subsequent chapters. In particular we define the, perhaps not so familiar, complex differential operator $\eth$ (edth) of Newman and Penrose.

In chapter three we develop the appropriate Sobolev-regularity theory for linear parabolic equations required to deal with the quasispherical initial data constraint equations. We include a result due to Polden [P] here, with a corrected proof. This result was essential for deriving the results contained in the later chapters of [P], and it is for this reason we include the result. We don't make use of it explicitly when considering the quasispherical Einstein constraints, but the ideas employed are similar to those we use to tackle the problem of existence for the quasispherical constraints.

Chapter four is concerned with the local existence of quasispherical initial data. We firstly consider the question of existence and uniqueness when the mean curvature of the 3-manifold is prescribed, then after introducing the notion of polar curvature, we also present another quasispherical constraint system in which we consider the polar curvature as prescribed. We prove local existence and uniqueness results for both of these alternate formulations of the quasispherical constraints.

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