On Spectrum of some $\boldsymbol{sl_2}$ and $\mathcal{U}_{\boldsymbol{q}}(\boldsymbol{sl_2})$ related Hamiltonians

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Abstract

We will deal with the representation theory of classical Lie Algebra and representation theory of quantum enveloping $\mathcal{U}_q(sl_2)$. The Lie algebra sl_2 and quantum algebra $\mathcal{U}_q(sl_2)$ are considered in a certain infinitely dimensional representation corresponding to "lowest weight 1". The representation module is equivalent to the Fock space representation of the quantum mechanical oscillator. The canonical elements e and f of sl_2, related to the "creatiion" and "annihilation" operators, appear to be anti-unitary in our construction, so that the operator $H = \frac{e+f}{\mathrm{i}}$ is Hermitian, and therefore it can be interpreted as a Hamiltonian for a certain Quantum Mechanical System. We will prove that this operator has continuous spectrum. Eigenstates of H are constructed explicitly. Our results are based on the representation of the fundamental solution set of the difference equation in the terms of slowly convergent semi-infinite matrix product and the analysis of its asymptotic.

A provision of the basic algebraic principles that help on easing the insight of this research and catching up the fundamentals of this thesis is incorporated. We expose some basics on Lie algebras and their representation theory, quantum mechanical aspects including the Fock space representation for quantum mechanical oscillator, and their relation to the representation theory of sl_2.

Also, we dialogue on quantum deformation of simple Lie algebras and their common features. At the end, we expose the category of modular representations of $\mathcal{U}_q(sl_2)$. Adoption of the developed technique of analysis of slowly convergent semi-infinite matrix products to the framework of modular-type difference equations will the next step of our study.

We sum up the overview of the thesis and we present major findings and similar studies. In conclusion, we summarise the major results of the research against the research questions investigated. An analysis of the major findings is presented.

Dedication

I lovingly dedicate this thesis to my parents for their endless inspiration and love through all my walks of life.
The most special thanks and gratitude go to my beloved wife for her essential support and continued encouragement.
This work is also dedicated to my princes Elaf and Wesam who have made me stronger, better and more fulfilled than I could have ever imagined. I love you to the moon and back.

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Publications from thesis

Here, we declare the main papers related to this thesis that has been realized by the author of this thesis oneself.

(1) Fahad M. Alamrani, "On eigenstates for some sl_2 related Hamiltonian", Acta Universitatis Apulensis 57 (2019), pp. 93-100, doi: 10.17114/j.aua.2019.57.08, under publication.

(2) Fahad M. Alamrani, "Representation Theory of sl_2, $\mathcal{U}_q(sl_2)$, oscillator, and q-oscillator", (2021), University of Canberra, under preparation.

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Chapter 1

Introduction

1.1 Primer

Representation theory has many interactions with other fields of mathematics, including noncommutative algebra, algebraic geometry, and mathematical physics, with applications in both directions, and constitutes a major area of current mathematical research [19]. Many aspects of the traditional theory of finite-dimensional semisimple Lie algebras, due to Killing et al. [20] and others around the turn of twentieth century [21], have found vast generalizations in recent decades in a number of different directions such as Kac-Moody algebras (especially affine Lie algebras) [22], vertex algebras and "quantum" [23] and "super" versions of these algebras.

This thesis will investigate representation theory of the classical Lie algebra [2] and quantum enveloping $\boldsymbol{\mathcal{U}_q(sl_2)}$. Within both representation theories, the Lie algebra $\boldsymbol{sl_2}$ is treated in a certain infinitely dimensional representation corresponding to the "lowest weight 1". This representation module is equivalent to the Fock space representation of the quantum oscillator [3], with non-standard "anti-unitary" feature, $\boldsymbol{e}^\dagger = -\boldsymbol{f}$, where $\boldsymbol{e}$ and $\boldsymbol{f}$ of the Cartan-Weyl basis of $\boldsymbol{sl_2}$ are seen as the "creation" and "annihilation" operators. Accordingly, the operator $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$ is Hermitian and it can be interpreted as a Hamiltonian for a certain quantum mechanical system[1]. Notably, this Hamiltonian is related to a Hamiltonian used in [4, 5] in the limit of $q = 1$. (Note that $q = 1$ regime was not considered in [4, 5]).

[1] An alternative choice, $\boldsymbol{H}' = \boldsymbol{e} - \boldsymbol{f}$, will be considered elswhere

1.2 Background

1.2.1 Standards of the quantum mechanics

In this background section, we will remind the basic concepts of the quantum mechanics. The aim is to provide an integrity of the thesis bringing relationships between theoretical physics and mathematics [1]. We split the listing of the concepts into two components. Firstly, we cover the basic mathematics framework, and then we inspect the principles of measurement theory.

Axioms of quantum mechanics

In classical physical sciences, any point in a "phase space" represents a state of a physical system [24]. We will certainly consider the space of solutions of an equation of motion, or as the space of potitions and energy (parametrising solutions by beginning esteem details). Observable quantities are considered being functions in this space; for example, the functions of coordinated and momenta. One noticeable evidence, the Hamiltonian (or energy), identifies just how states advance in time with equations of Hamilton equations [27].

The basic framework of quantum mechanics is extremely unique [28] and its formalism is built on the following straightforward axioms:

Axiom (States): *A state of a quantum system is given by a nonzero vector* $|\psi\rangle$ *in a complex vector space* $\mathbb{H}$ *with Hermitian inner product* $\langle\overline{\psi}_1, \psi_2\rangle$ [25].

Mathematically, we deal with the notions of some linear algebra, such as products' properties in complex vector spaces. $\mathbb{H}$ might be infinite or finite-dimensional along with some conditions needed within the infinite-dimensional situation; for example, we may have to ensure that $\mathbb{H}$ is a Hilbert space [6] . We record two essential contrasts from states of classical mechanical system:

First, linearity of the state space, i.e., that a linear combination of states gives also a state. Second, the state of space is a *complex* vector space, i.e., that linear combinations significantly include complex numbers, which come alongside complex numbers utilised as necessary calculational tool.

We will consider Dirac notation [26] in the state space $\mathbb{H}$ for vectors where a vector is labeled "ψ" as follows:

$$|\psi\rangle$$

Axiom (Observables): *There is the observables of a quantum mechanical system, which can be provided by self-adjoint linear operators on* $\mathbb{H}$.

Axiom (Dynamics): *The prominent evident is the Hamiltonian* $\boldsymbol{\mathcal{H}}$*, where that the time evolution of states* $|\psi(t)\rangle \in \mathbb{H}$ *is provided by the non-stationary Schrödinger equation*

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\boldsymbol{\mathcal{H}}|\psi(t)\rangle \tag{1.2.1.1}$$

We use a physical analysis to the Hamiltonian evident $\boldsymbol{\mathcal{H}}$ in regards to energy [27]. We might as well require to define some sort of positivity characteristic on $\boldsymbol{\mathcal{H}}$, in order to ensure the least steady energy state.

The dimensional constant is "$\hbar$" and also its worth relies on kind of units utilises for energy and time. It reveals the measurements $[time] \cdot [energy]$ and its exploratory values such as:

$$1.054571726(47) \times 10^{-34}\ \mathrm{J} \cdot \mathrm{s} = 6.58211928(15) \times 10^{-16}\ \mathrm{eV} \cdot \mathrm{s} \tag{1.2.1.2}$$

(eV is represented by the unit of "electron-Volt". The electron's energy should be obtained by relocating on the one-Volt electric potential). Quantum mechanical concerns make use of natural units of energy as well as time of selected units, therefore that $\hbar = 1$. For event, we utilise seconds regard to measure energies and time during little units of 6.6 x 10^{-16} eV; other option is eV can be utilised for energies, and after that, the little units for 6.6×10^{-16} seconds regard to time. If one is looking at the system, then that will indicate Schrödinger's equation [7], where the representative energy scale is defined by eV and the state vector will be variable through exceptionally brief time scale of 6.6×10^{-16} seconds. We, as a rule, set $\hbar =$

1, verifiably reaching to a unit natural behaviour for quantum mechanics. The ultimate result gives the plausibility to embed suitable factors of $\hbar$ so as to get solutions in more classical unit systems.
Nevertheless, in many cases, it is valuable to put together factors of $\hbar$. Doing so, offer supports on clearing up which terms contrast to classical physics system, and that are definitely quantum mechanical behaviour. On regular basis, classical physics in nature concerns the limit, then

$$\frac{(energy\ scale)(time\ scale)}{\hbar} \tag{1.2.1.3}$$

is significant. Commonly authentic for time scales and the energy experienced in requirement of living, however it may furthermore be accomplished by taking into consideration that $\hbar \to 0$, which mentions to the "classical limit". We have to be beyond any doubt the way through which classical system develops out of quantum theory aspect like a limit which may be an extremely intricate phenomenon.

Fundamental portions of measurement theory

The mathematical framework of a quantum theory is characterised by the above axioms; however, they do not deal with the "measurement trouble". That concern of measurement inquiries just how to use this mathematical framework to a physical structure engaging with some type of macroscopic and some sort of human-scale exploratory tool. This irritable problem requires for the contemplate of two communicating quantum behaviours (the measurement apparatus and also the one being measured), not only in a total product of the two states, but extremely "entrapped". A macroscopic device consists of something like 10^{23} levels of flexibility, and it reaches to be incredibly difficult to evaluate inside the part of quantum mechanical system (needing, for example, the arrangement of the Schrödinger equation with 10^{23} variables).

The issue of just how macroscopic classical physics system increases in a measurement procedure, may be deal with based on two concepts just as a phenomenological portrayal of what shall certainly occur, which will provide specific statistical prognosis using quantum theory side:

Concept. *A state for which an observable quantity is uniquely defined real number is an eigenstate of a corresponding self-adjoint operator.*

This guideline acknowledges the states where we will interface smartly a label to the eigenvalue, that represents an evident amount identifying states in classical mechanics. Angular momentum, energy, momentum, or charge speak to the primary used noticeable and represent some sort of group action on the physical behaviour.

Principle (The Born rule). *Given an observable (self-adjoint operator) O and two normalised states $|\psi_1\rangle$ and $|\psi_2\rangle$,*

$$\langle\overline{\psi}_i|\psi_j\rangle \;=\; \delta_{i,j} \;, \tag{1.2.1.4}$$

that are the eigenvectors of operator O with eigenvalues λ_1 and λ_2

$$O|\psi_1\rangle = \lambda_1|\psi_1\rangle, \quad O|\psi_2\rangle = \lambda_2|\psi_2\rangle \tag{1.2.1.5}$$

then the complex linear combination

$$|\psi\rangle \;=\; c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{1.2.1.6}$$

may not have a well-defined value for the observable O. If one attempts to measure this observable, one will get either λ_1 or λ_2, with probabilities

$$\frac{\langle\overline{\psi}|\psi_1\rangle\langle\overline{\psi}_1|\psi\rangle}{\langle\overline{\psi}|\psi\rangle} \;=\; \frac{|c_1^2|}{|c_1^2| \;+\; |c_2^2|} \tag{1.2.1.7}$$

and

$$\frac{\langle\overline{\psi}|\psi_2\rangle\langle\overline{\psi}_2|\psi\rangle}{\langle\overline{\psi}|\psi\rangle} \;=\; \frac{|c_2^2|}{|c_1^2| \;+\; |c_2^2|} \tag{1.2.1.8}$$

respectively.

The Born rule [8] might be elevated to the level of theory of an axiom, where that is offered a complete realisation of exactly how measurements function and the even more essential axioms of the previous part. The obstacle is just how classical system increases in tests with the suggestion of "decoherence".

Keep in mind that the state $c|\psi\rangle$ shall obtain the exact same worth of the eigenvalues and the probabilities as the state $|\psi\rangle$, for any type of complex number c. It is standard to collaborate with states having a settled standard worth of 1, that describes the amplitude of c, taking off a residual phase $e^{i\theta}$ equivocalness. Consenting to more than criteria, we then neglect this phase details which will not add to a computed probabilities of measurements.

1.2.2 Theory of unitary group representations

The mathematical methods of quantum mechanics are related to the theory of "unitary group representations"[9]. Here could give a short overview of the appropriate definitions, and a sign of the partnership to the quantum theory area.

Group. *A set G is called a group if it is equipped by an associative multiplication with the unity:*

$$\forall\, g_1, g_2 \in G \quad \exists\; g_3 \in G \;:\; g_1 \cdot g_2 \;=\; g_3 \;, \quad g_1(g_2 g_3) \;=\; (g_1 g_2) g_3 \;, \quad 1 \in G \;.$$

Among various mathematical groups of rate of interest, we are mosting likely to be captivated by the group of all conversions regarding a class of 3-dimensional space. The majority of groups are taken in consideration as "matrix groups" that are famous as subgroups of the group of n by n invertible matrices (with complex or real matrix elements). In this instant, the group multiplication is just the matrix multiplication.

Representation: *The representation* (π, V) *of a group G is a homomorphism*

$$\pi : g \in G \to \pi(g) \in GL(V)$$

where the group of invertible linear maps $V \to V$ with V a complex vector space, is $GL(V)$.

The map of π is a homomorphism [10], what means

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2) \tag{1.2.2.1}$$

where that for both $g_1, g_2 \in G$. We will acknowledge linear maps and matrices, then that shall give us an isomorphism when V is finite dimensional and we have a basis of V, that provides us with an isomorphism

$$GL(V) \simeq GL(n, \mathbb{C}) \tag{1.2.2.2}$$

where $GL(n, \mathbb{C})$ is the group of invertible n by n matrices with complex matrix elements. We start with representations which should be finite dimensional ones and effort to produce comprehensive declarations. After that, we sight at representations at the function spaces, that are infinite dimensional part, and also take into consideration the authentic informative obstacles which emerge when we attempt to create mathematically accurate declarations within the situation of infinite-dimensional .

The representations of (π, V) suggest in many cases to the map π, taking off implied the vector space of V and, other times, as well as the matrices of $\pi(g)$ act on, they describe by defining the vector space of V, cleaning out comprehended the map of π. It is obvious that one cause for this may be that the map of π can be the identity map, where that frequently G illustrates as kind of a matrix group, therefore a subgroup of $GL(n,\mathbb{C})$, acting upon $V \simeq \mathbb{C}^n$ by the conventional task of matrices on the vectors. It should certainly be beyond a shadow of doubt, which simplify defining V, is for the essentially part not sufficient to define the representation, because it can not be the conventional one, as an example, it maybe undoubtedly the insignificant representation at V, as following:

$$\pi(g) = 1_n \tag{1.2.2.3}$$

such as every element of G appear at V as the identity.

In mathematical aspect, the primary oddly courses of complex representations originate from the significant of the linear transformations of $\pi(g)$ which are "unitary", maintaining the concept of size offered by the common Hermitian inner of product, therefore , denoting the vectors of unit to the vectors of unit. For physical side, the group representations underneath assumed on regular basis compatible with physical symmetries concepts, and unitary transformations of a Hamiltonian $\boldsymbol{\mathcal{H}}$ will certainly leave regular the probabilities of diverse observances.

Unitary representation. *A representation is unitary if it preserves the inner* *product. It is a representation* $(\pi,\ V)$ *on a complex vector space* V *with Hermitian inner product* $\langle\cdot,\cdot\rangle$, such as

$$\langle \pi(g)v_1, \pi(g)v_2\rangle = \langle v_1, v_2\rangle \tag{1.2.2.4}$$

for all $v_1, v_2 \in V$ *and* $g \in G$ *(Hermitian conjugation here is implicitly assumed).*

The matrix $\pi(g)$ contains values in a subgroup $\boldsymbol{\mathcal{U}}(n) \subset GL(n,\mathbb{C})$ in a unitary representation. In the linear algebra portion, we will find that $\boldsymbol{\mathcal{U}}(n)$ might be identified by a group of n by n complex of matrices $\boldsymbol{\mathcal{U}}$ such that

$$\boldsymbol{\mathcal{U}}^{-1} = \boldsymbol{\mathcal{U}}^{\dagger} \tag{1.2.2.5}$$

where, as it is known, $\boldsymbol{\mathcal{U}}^{\dagger}$ represents the conjugate-transpose of $\boldsymbol{\mathcal{U}}$. Stamp that we are most likely to be making use of the symbol "†" to imply the "adjoint" or conjugate-matrix of transpose (Hermitian conjugation). This symbol is magnificently extensive in physical area, despite of mathematical field leawn toward to use "*" rather than "†".

1.2.3 Structure of quantum mechanics and representations

There is a basic partnership among quantum mechanics and representation theory that is the space of states $\mathbb{H}$ which will hand down a unitary representation of G (at tiniest as much as a phase of factor) at whenever possible point we got a physical quantum tool with a group G performing with it. The quantum mechanics, which deals with physical branch, indicates that representation theory supplies details about quantum mechanical state spaces. In mathematical aspect depiction representation theory, this suggests physics can be an extremely worthwhile resource of unitary representations to investigate (any physical behaviour with a symmetry group G shall supply one).

For group elements g, as well as a representation π that are close to the identity, it might make use of exponentiation to compose $\pi(g) \in GL(n,\ \mathbb{C})$ like that

$$\pi(g) = e^{A} \tag{1.2.3.1}$$

when $\pi(g)$ can be unitary, such as in subgroup $\boldsymbol{\mathcal{U}}(n) \subset GL(n, \mathbb{C})$, thus A shall represent skew-adjoint:

$$A^{\dagger} = -A \tag{1.2.3.2}$$

where the conjugate-transpose matrix is $A^{\dagger}$. When we define $B = iA$, then it is clear to us that B is self-adjoint:

$$B^{\dagger} = B \tag{1.2.3.3}$$

The instance of finite-dimensional $\mathbb{H}$, a unitary representation π of G on $\mathbb{H}$ originating from the symmetry G of our physical behaviour, provides us not only the unitary matrix $\pi(g)$, yet additionally matching self-adjoint operators B at $\mathbb{H}$. It is deduced that the symmetries supply a brand of quantum mechanical visible beside the self-adjointness property of these operators representing the reality that the symmetries are recognised as the unitary representations in the state space. Agreeing in this manner, this may be a shocking fact that for several physical behaviours the type of perceptible incorporates the among majority of physical rate of interest.

There are numerous instances of this phenomenon like the essential situation of "time-translation symmetry". We obtain a unitary representation of $\mathbb{R}$ on the space of states $\mathbb{H}$, as well as the group is $G = \mathbb{R}$ (with the additive group law) . Hence, the Hamiltonian operator $\boldsymbol{\mathcal{H}}$ (divided by $\hbar$) is the corresponding self-adjoint operator, and the representation is provided by

$$t \in \mathbb{R} \rightarrow \pi(t) = \exp\left(-\frac{i}{\hbar}\boldsymbol{H}t\right) \tag{1.2.3.4}$$

that is the group homomorphism from the group $\mathbb{R}$ to the group of unitary operators which one can inspect. The dynamics of the theory is shown by this unitary representation, the Schrödinger formula providing that the declaration $\frac{i}{\hbar}\boldsymbol{\mathcal{H}}\Delta t$ is the skew-adjoint operator that exponentiated to supply the unitary transformation that moves states $\psi(t)$ by a quantity $\Delta(t)$ ahead in time.

1.2.4 Concepts of representations on function spaces and their symmetry groups

Highlight the phenomenon of invariance of features of objects under sets of transformations that form a group, that is common to suggest the groups that turn up in this area as the symmetry groups. This may be a little bit tricking even with the reality, because we are not as it were interested on invariance, however the phenomenon of groups performing on sets.

Group action on a set: *The action of a group G on a set M is provided by the following map*

$$(g,\ x) \in G \otimes M \rightarrow g \cdot x \in M \tag{1.2.4.1}$$

It maps (g, x) of a group element $g \in G$ as well as an element $x \in M$ to an another element $g \cdot x \in M$. In particular,

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \tag{1.2.4.2}$$

It is noteworthy for us that the 3-dimensional space $M = \mathbb{R}^3$ with the standard inward product is a terrific illustration to be beyond a shadow of a doubt. This includes with two varied group actions preserving the inner product

- $G = \mathbb{R}^3$ on $\mathbb{R}^3$ by translations is the action of the group
- $\mathbb{G} = O(3)$ of 3-dimensional orthogonal transformations of $\mathbb{R}^3$ is the action of the group, where these are the transformations regarding the origin (certainly incorporated together with a reflection). Keep in mind for this situation, order issues that regarding non-commutative groups such as $\mathbb{G}$, one has $g_1 g_2 \neq g_2 g_1$ for some of group elements g_1, g_2.

Now days, it is known that mathematical essential concept reveals that the method to obtain a space of M, provided a little set of points, relies on to check with $F(M)$, the set functions at this space. For the reason the function space speaks to a vector of space, no issue regardless of what the geometrical framework of the initial set, then this "linearizes" the issue. The function space shall be a finite dimensional vector space, if our distinct set properties the finite number of elements. Thereafter, it can be the infinite dimensional as well as it will have to even more suggest the space of functions, such as continuous of functions, etc.

A group action of G is provided on M, when we take the complex functions on M, then it will supplies a representation $(\pi,\ F(M))$ of G, with π defined on functions f by

$$(\pi(g)f)(x) = f(g^{-1}\cdot x) \tag{1.2.4.3}$$

It is remarkable that the demanded inverse to motivate the group of homomorphism feature for cooperative, due to one holds

$$\begin{aligned}(\pi(g_1)\pi(g_2)f)(x) &= (\pi(g_2)f)(x)(g^{-1}\cdot x)\\ &= f(g_2^{-1}\cdot(g_1^{-1}\cdot x))\\ &= f((g_2^{-1}g_1^{-1})\cdot x)\\ &= f((g_1g_2)^{-1}\cdot x)\\ &= (\pi(g_1g_2)f)(x)\end{aligned} \tag{1.2.4.4}$$

Therefore, that computation certainly would not practice appropriately for non-commutative of G if one specified such as $(\pi(g)f)(x) = f(g\cdot x)$.

The reasonable manner can make development of quantum mechanical of state spaces $\mathbb{H}$ passes by the wavefunctions, indicating the complex-valued of functions at space of time. The representation π on the state space $\mathbb{H}$ of such the wavefunctions can be obtained for provided any type of group action on space of time.

1.2.5 Origin of quantum groups

Within the early 1980's, one curiously issue was understanding of exactly solvable models in quantum mechanics that includes integrable systems. The quantum Yang-Baxter equation [30][31] and the quantum inverse scattering method [29][30] are two key tools for this area of study. Briefly, a way of finding correct arrangements of two-dimensional models in quantum area theory and statistical physics is the quantum inverse scattering technique [29][30]. In spite of the fact that the inverse scattering was major to the advancement of the quantum groups field, subtle elements are a little bit overwhelming physics.

From their beginning and inception, the quantum groups have included within their physics nursery to distant coming to impacts in immaculate the area of mathematics. For occasion, many fields were impacted by the quantum group like representation, topology, analysis, combinatorics, non-commutative geometry, symplectic geometry, and knot theory to title some [50]. The first discovered quantum group was the q-analogue of $SU(2)$ and it was not famous as such. In 1985, Drinfel'd created Quantum group at the side with M. Jimbo, did wide job within the part of integrable systems [32]. To begin with, it was figured that quantum groups to be associative algebras whose characterising relations, are communicated in conditions of a matrix of constants known as a quantum R-matrix, where universal R-matrix are additionally ascribed to Drinfel'd. These algebras are truly Hopf algebras [33] autonomously watched by Drinfel'd and Jimbo, where that happened in 1985 as well. Hopf algebras themselves were presented in the 50's despite they were not novel at this time and have been inspected in profundity since the 60's . The language of Hopf algebras is more than altogether valuable, but the critical highlight of this specific type of Hopf algebras is that they stand for contortions of all inclusive wrapping algebras of classical Lie algebras as well as the matrix groups [37]. Behind the term of *quantum* group, it takes after the concept of quantum mechanics as a distortion of the classical mechanics. At the side ground breaking examples at the International Congress of Mathematics in 1986, this new object was presented by Drinfel'd. After that, there were non-commutative distortions of the algebra of functions on $SL_2(\mathbb{C})$ and $SU(2)$ which freely developed by Manin [34] and Woronowicz [35].

These distortions were initially aiming to assist the solutions of development to the presently popular Yang-Baxter equation [30][31]. The Yang-Baxter equation braces significant value to cutting edge area of theoretical physics . Actually, it may essentially give the premise of quantum group theory [11][38], considering Yang-Baxter equation's solutions [49] that supply a beginning point for the quantum inverse scattering technique [12][29][36]. As said over, it driven to the detection of quantum groups in the first area. Nowadays, we accept quantum groups which offer the essential framework for dealing with the holy grail of physics, such as the unification of the quantum mechanics with gravity, where that makes quantum groups an appealing and motivating region of study.

1.3 Research questions

We begin the development of this thesis by introducing algebraic basis of quantum mechanics. We figure out Lies algebras and their representation theory. We spell out the representation theory in classical algebra. We scrutinise the representation theory of $\boldsymbol{sl_2}$ and $\boldsymbol{\mathcal{U}_q(sl_2)}$, investigate the oscillator, and q-oscillator, and inspect the Hamiltonians related to $\boldsymbol{sl_2}$.

In this thesis, we prompt some research questions that we attempt to behave, if not, at least paving the way for new discoveries. Hence, we ask for: What is the suitable representation of $\boldsymbol{sl_2}$? What is the stationary Schrödinger equation as a linear recursion with non-constant coefficients? How can we analyse our recursion? How we define $\delta(n, E)$ and $\epsilon(n, E)$ analytically in the forms of series expansion with respect to $1/n$ and E? And, what is the asymptotic of our recursion? (See chapter 3 for the details)

Additionally, we arouse queries such as: How to formulate the occupation number operator? How to gauge the well-known exchange relations between the three elements $\boldsymbol{e}$, $\boldsymbol{f}$, and $\boldsymbol{h}$ of the representation theory of algebra $\boldsymbol{sl_2}$ through the construction of right and left module and the application of quantum mechanical approach? How to probe the essential notations for the q-deformation of $\boldsymbol{sl_2}$? How to establish the homomorphism from the q-oscillator algebra $\mathcal{O}_q$ and its Fock space representation via the change of the normalisation of states and $\lambda_0 = 1$ with spin = -$\frac{1}{2}$ representation of $\boldsymbol{\mathcal{U}_q(sl_2)}$? What is the alternative of the Schrödinger problem for the quantum deformed algebras? What is the advantage of q-deformation? How to briefly describe another class of representations of $\boldsymbol{\mathcal{U}_q(sl_2)}$ using that $\boldsymbol{u}$ and $\boldsymbol{v}$ be the generators of the simple Weyl algebra?

By grappling with these research inquiries, we herd us to hunt for developing new research projects and induce the opportunity to further progress within this exciting world of achievable auspicious theory that could guide us to outstanding discoveries.

1.4 Thesis outline

In Chapter 1, we introduce the basic principles that one needs to know to understand this thesis. We will investigate representation theories of the classical Lie algebra and quantum group $\boldsymbol{\mathcal{U}_q(sl_2)}$. In the background section, we introduce the fundamental principles of quantum mechanics that drive this thesis. We look at the states and observables of quantum mechanics axioms. We also discuss the observables and Born rule [6] principles of measurement theory. We then present the fundamental relationship between representation theory and quantum mechanics. We analyse the theory of unitary group representations and the symmetry groups representations on function spaces. We finally take a look at quantum groups solvable models.

In Chapter 2, we study the Lies algebra and their representation theory. We begin by presenting the group $SO(3)$ and its algebra. We then have a section where we look at the angular momentum in quantum mechanics, including Pauli matrices and nicknames, Casimir operator, the basic in quantum mechanic, and the fields in quantum theories. Thereafter, we provide examples and explanation of Fock space of $\boldsymbol{sl_2}$ representation theory. We integrate a section on simple lie algebras and their common features, the algebras of $\boldsymbol{sl_n}$, and the Cartan-Weyl basis theorem. It also includes a section on the construction of the representation theory of simple Lie algebra.

Chapter 3 concerns eigenstates for some $\boldsymbol{sl_2}$ related Hamiltonian. Section 2 is the formulation of the problem, which is the proper representation of $\boldsymbol{sl_2}$, in addition to rewrite the stationary Schrödinger equation as a linear recursion with variable coefficients. Section 3 shows the analysis of the corresponding recursion equations. Sections 4 and 5 represent the orthogonality and concluding discussions, respectively.

Chapter 4 introduces $\boldsymbol{\mathcal{U}_q(sl_2)}$ and their representations. The primary objective of this chapter is to demonstrate that the new algebra shares two main properties with the old one: The basic in quantum mechanic, and it has no zero divisors. This implies that we have to derive commutator formulas.

Chapter 5 investigates the oscillator algebra and the Fock space, the Representation theory of $\boldsymbol{sl_2}$ using quantum mechanic, $\boldsymbol{\mathcal{U}_q(sl_2)}$ and the q-oscillator, the Hamiltonian in $\boldsymbol{\mathcal{U}_q(sl_2)}$ by reformulating the Schrödinger problem for the quantum deformed algebras, and we end by a modular of another representation of $\boldsymbol{\mathcal{U}_q(sl_2)}$.

Chapter 6 presents the discussion of the thesis through a recall summary of each chapter, the presentation of the major findings, a discussion over similar studies and the formulation of some recommendations for further research.

Chapter 7 encompasses the general conclusion of the thesis and sets up the table for challenging future similar related problems.

Chapter 2

Lie algebras and their representation theory

2.1 Introduction

We outline some basic principles concerning the Lie algebras and their representation theory in this chapter. This chapter also has a rather scholar nature, it is based on an advanced teaching course [65] to which I was involved.

The basic principles relating the group theory and their algebras are demonstrated on the example of orthogonal rotation group $SO(3)$ [39]. We look over Angular momentum in quantum mechanics [3][9] together with Pauli matrices and the Casimir operator [13]. We talk about in the spirit of quantum mechanics and the areas in quantum hypotheses [24][30][33][34]. We see at a few cases, Fock Space [14] and a general case within the representation theory [2]. At that point we audit a few highlights of the simple Lies algebras beside Bianci character and Cartan-Weyl theorem.

2.2 Group $SO(3)$ and its algebra

The invariance of scientific study enactments of physics with regard to a selection of a frame of reference is originated from mathematical concept of a group. Specifically, the option of the basis can not be relied upon the geometry of Euclidean space.

Let the orthogonal basis of n-dimensional Euclidean space E_n be $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$.

The Pythagoras theorem is equivalent to the matrix of scalar products.

$$(\boldsymbol{e}_j, \boldsymbol{e}_k) \;=\; \delta_{j,k} \tag{2.2.1}$$

For any vector at E_n there is a unique decomposition

$$\mathbf{x} \;=\; x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n \tag{2.2.2}$$

The components of vector $\mathbf{x}$ relative to the provided basis $\mathbf{e}_i$ are coefficients x_i .

Suppose that $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_n$ denote another orthogonal basis of E_n obtained by a *smooth* rotation of the initial basis. From (2.2.2), it might reveals $\boldsymbol{e}'_j$ regard to $\boldsymbol{e}_j$:

$$\boldsymbol{e}'_j \;=\; \sum_k u_{j,k}\boldsymbol{e}_k \quad \text{or, in matrix form,} \quad \boldsymbol{e}' = u \cdot \boldsymbol{e} \;, \quad u = ||u_{j,k}|| \tag{2.2.3}$$

The terms of orthogonality (2.2.1) with the new basis imply

$$(\boldsymbol{e}'_j, \boldsymbol{e}'_k) = \sum_{l,m}(u_{jl}\boldsymbol{e}_l, u_{km}\boldsymbol{e}_m) = \sum_{lm} u_{jl}u_{km}\delta_{l,m} = \sum_m u_{jm}u_{km} = \delta_{jk} \;\Leftrightarrow\; u^T \cdot u = \mathbb{1} \tag{2.2.4}$$

where $\mathbb{1} = ||\delta_{jk}||$ is the identity matrix.

According to physical laws, that deem the changes of components of a vector, there are side-to-side correspondence among the change of the basis and change of coordinates:

$$\begin{aligned} \mathbf{x} = \sum_k x_k\boldsymbol{e}_k \equiv \sum_j x'_j\boldsymbol{e}'_j = \sum_k\left(\sum_j x'_j u_{jk}\right)\boldsymbol{e}_k \\ \Rightarrow\; x_k = \sum_j x'_j u_{jk} \quad \text{or} \quad x'_j = \sum_k u_{jk}x_k \end{aligned} \tag{2.2.5}$$

Clearly, the changes of the coordinates x_j and the changes of the basis $\boldsymbol{e}_j$ concur that this is always a specific characteristic of Euclidian metric.

The basis $\boldsymbol{e}'_j$ is mentioned as a result of a smooth alternation of the basis $\boldsymbol{e}$, where the basis $\boldsymbol{e}'_j$ can be rotated and acquired the third basis $\boldsymbol{e}''_j$. When basis $\boldsymbol{e}''$ is broken down relative to $\boldsymbol{e}'$, $\boldsymbol{e}'' = u_2 \cdot \boldsymbol{e}'$, and $\boldsymbol{e}'$ is decomposed with respect to $\boldsymbol{e}$, $\boldsymbol{e}' = u_1 \cdot \boldsymbol{e}$, then $\boldsymbol{e}'' = u \cdot \boldsymbol{e}$, $u = u_2 \cdot u_1$. The orthogonality can be preserved with each action:

$$u_1^T \cdot u_1 = \mathbb{1} \;, \quad u_2^T \cdot u_2 = \mathbb{1} \;\Rightarrow\; (u_2u_1)^T \cdot (u_2u_1) = u_1^T u_2^T u_2 u_1 = \mathbb{1} \;. \tag{2.2.6}$$

This shows the framework of the matrix group:

(1) A set G is called the (matrix) group if (1) it is defined the product on G, namely for any $u_1, u_2 \in G$ their product $u_1 \cdot u_2 \in G$, and (2) there is the identity element $\mathbb{1} \in G$: $u \cdot \mathbb{1} = \mathbb{1} \cdot u = u$, at any $u \in G$, and (3) for any element there is its inverse: for any $u \in G$ there is $u^{-1} \in G$ such that $u \cdot u^{-1} = u^{-1} \cdot u = \mathbb{1}$. G is called the matrix Lie group when it is a smooth variety

The special orthogonal group $SO(n)$ is the set of n by n matrices u fulfilling the terms

$$u^T \cdot u = \mathbb{1} \quad \text{and} \quad \det(u) = 1 \tag{2.2.7}$$

The last requirement, $\det(u) = 1$, just prohibits the reflections and makes the group variety simply-connected.

Now, move to some illustrations. E_2 is the most basic situation, where at the transformation of the structure $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ through the angle θ the Pythagoras theory betokens

$$\begin{aligned} \boldsymbol{e}_1^{'} &= \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2 \\ \boldsymbol{e}_2^{'} &= -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2 \end{aligned}$$

The matrix form reads

$$\boldsymbol{e}' = u(\theta) \cdot \boldsymbol{e} \;, \qquad u(\theta) \;=\; \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.2.8}$$

It is clear that $u(\theta)^T u(\theta) = \mathbb{1}$ and $\det[u(\theta)] = 1$. Then, the group of multiplication implies $u(\theta)u(\theta') = u(\theta + \theta')$. Thus, $u(\theta)$ is in fact an exponent.

The matrix of exponent is indicated by Taylor series expansion

$$\mathrm{e}^A \stackrel{\text{def}}{=} \mathbb{1} + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \cdots \tag{2.2.9}$$

Cosine and sine as power series of using the growth of exponent and definitions, one will clarify

$$u(\theta) = \mathrm{e}^{\theta J} \;, \quad \text{where} \quad J \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;, \quad J^2 = -\mathbb{1} \tag{2.2.10}$$

To acquire the matrix J is to let $u(\theta)$ for small θ: $\cos\theta = 1 - \theta^2/2 + \ldots$ and $\sin\theta = \theta - \theta^3/6 + \ldots$, then $u(\theta) = 1 + \theta J + \ldots$.

After that, transform to E_3 case. The specific parameterisation of the component u (2.2.3) of orthogonal three-dimensional of group in the conditions of Euler angles ψ, θ, ϕ equal to a deterioration of u at straightforward alternations (2.2.8):

$$u = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}}_{u_1(\psi)} \cdot \underbrace{\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}}_{u_2(\theta)} \cdot \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{u_3(\phi)} \tag{2.2.11}$$

respecting (2.2.10), we can check this matrix at exponential shape,

$$u_1(\psi) = \mathrm{e}^{\mathrm{i}\psi T_1} \underset{\psi\to 0}{\simeq} \mathbb{1} + \mathrm{i}\psi T_1, \;\; u_2(\theta) = \mathrm{e}^{-\mathrm{i}\theta T_2} \underset{\theta\to 0}{\simeq} \mathbb{1} - \mathrm{i}\theta T_2, \;\; u_3(\phi) = \mathrm{e}^{\mathrm{i}\phi T_3} \underset{\phi\to 0}{\simeq} \mathbb{1} + \mathrm{i}\phi T_3 \;, \tag{2.2.12}$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2.13}$$

The Euler deterioration in arrange was used to liberate the matrices T_j generating the revolutions around j^{th} tomahawks. Another address is just how to multiply the exponents at $u = \mathrm{e}^{\mathrm{i}\psi T_1}\mathrm{e}^{-\mathrm{i}\theta T_2}\mathrm{e}^{\mathrm{i}\phi T_3}$ and how to obtain the group of multiplications from the viewpoint of exponents?
The answer for that is an analytical formulation – the Baker Campbell-Hausdorf identity[1][40] [41].

Define the matrices which get an exponential form, $u_1 = \mathrm{e}^A$ and $u_2 = \mathrm{e}^B$. At that spot their product may be as

$$\begin{aligned} \mathrm{e}^A \cdot \mathrm{e}^B &= \mathrm{e}^{C(A,B)} \;, \\ C(A,B) &\equiv A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\Big([[A,B],B] + [A,[A,B]]\Big) - \tfrac{1}{720}\cdots \end{aligned} \tag{2.2.15}$$

[1]More famous Baker-Campbell-Hausdorf identity is

$$\mathrm{e}^A \cdot B \cdot \mathrm{e}^{-A} = B + \sum_{n=1}^{\infty} \frac{1}{n!}[\underbrace{A,[A,\ldots[A}_{n\text{ times}},B]\ldots]] \tag{2.2.14}$$

$[A, B]$ should be commutator:

$$[A, B] \stackrel{\text{def}}{=} AB - BA \tag{2.2.16}$$

The infinite series represent expression for $C(A, B)$. According to (2.2.15) is that all the summands next $A + B$ are a commutator, then the commutators of the commutators, and so on.

Let us check at that point, the nature of what the matrices create: the commutators of the generators which is defined by T_j. Simply

$$[T_1, T_2] = \mathrm{i}T_3 \ , \quad [T_2, T_3] = \mathrm{i}T_1 \ , \quad [T_3, T_1] = \mathrm{i}T_2 \ , \tag{2.2.17}$$

The set of T_j can be closed beneath commutation, for that

$$u \ = \ \mathrm{e}^{\mathrm{i}\psi T_1 - \mathrm{i}\theta T_2 + \mathrm{i}\phi T_3 + \frac{\mathrm{i}}{2}\theta\phi T_1 + \frac{\mathrm{i}}{2}\psi\phi T_2 + \frac{\mathrm{i}}{2}\psi\theta T_3 + \cdots} \ = \ \mathrm{e}^{\mathrm{i}(x_1 T_1 + x_2 T_2 + x_3 T_3)} \tag{2.2.18}$$

Furthermore, $u_1 \cdot u_2$ would have closely resembling exponential framework when the group matrices u_1 and u_2 have like exponential framework, due to the BCH identity [40] [41] and because the set of T_j is just closed beneath commutation. The notion of Lie algebra [42] can be given by the logarithm of (2.2.15) or (2.2.18):

> The Lie algebra when A and B from $\mathfrak{L}$,
> (1) is a set of operators $\mathfrak{L}$, their linear combination $aA + bB$ belongs to $\mathfrak{L}$ (a, b are numbers),
> (2) their commutator $[A, B]$ as well belong to $\mathfrak{L}$.

Hence,the first feature is that the Lie algebra should be a sort of *linear space* [43], for instance, selecting the basis $\mathcal{J}_\alpha$, $\alpha = 1, \ldots, n$ where n is denoted the dimension of algebra. If the basis is selected, then the algebra can be defined by the commutation laws

$$[\mathcal{J}_\alpha, \mathcal{J}_\beta] \ = \ \sum_\gamma c_{\alpha\beta;\gamma} \mathcal{J}_\gamma \tag{2.2.19}$$

The structure of constants are the set of numbers $c_{\alpha\beta;\gamma}$. The two mentioned essential features of the commutator: its anti-symmetry and Jacobi identity [44]:

$$[A, B] \ = \ -[B, A] \quad \text{and} \quad [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2.2.20}$$

It is known for us that T_1, T_2, T_3 is the basis, as well as (2.2.17) can be appeared in (2.2.19) shape:

$$[T_i, T_j] \ = \ \sum_k \ \mathrm{i}\epsilon_{ijk} T_k \tag{2.2.21}$$

where $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric "*tensor*" with $\epsilon_{123} = +1$.

2.3 Angular momentum aspect

2.3.1 Angular momentum in quantum mechanics

Regarding to the mechanics field, any inner symmetry in a mechanical framework generates a moderating amount, the integral of motion. In the event that the mechanical behaviour usually is invariant with regard to turns [it is the situation of the focal of power], matching moderating amount is the angular momentum vector [45]. If there exist, the system acquires the unique tool with the following coordinate $\mathbf{x} = (x_1, x_2, x_3)$ and also momentum value $\mathbf{p} = (p_1, p_2, p_3)$, then the components angular momentum $\mathbf{M} = \mathbf{x} \times \mathbf{p}$ are

$$M_i = \sum_{j,k} \epsilon_{ijk} x_j p_k \ : \quad M_1 = x_2 p_3 - x_3 p_2 \ , \quad M_2 = x_3 p_1 - x_1 p_3 \ , \quad M_3 = x_1 p_2 - x_2 p_1 \tag{2.3.1.1}$$

In quantum mechanical structure, both the momenta and the coordinates should be operators:

$$[x, p] \ = \ \mathrm{i}\hbar \ , \quad [x_j, p_k] = \mathrm{i}\hbar \ \delta_{j,k} \tag{2.3.1.2}$$

The quantum angular momentum vector of components [45][61] turn into operators, where they are as yet preserving amounts, they transfer with the Hamiltonian of a behaviour with a central field potential $\boldsymbol{\mathcal{H}} = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x}^2)$. However, let us look at commutators[2] for $L_j = \frac{M_j}{\hbar}$:

$$[L_1, L_2] = \mathrm{i}L_3 \ , \quad [L_2, L_3] = \mathrm{i}L_2 \ , \quad [L_3, L_1] = \mathrm{i}L_1 \tag{2.3.1.5}$$

Comparing with (2.2.17), we see variable mathematical base of the quantum angular of momentum concurs with the algebra rotation group [46]. The following notions are essential of quantum theories:

[2]Useful hints for calculations:

$$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[BC] + [AC]B \tag{2.3.1.3}$$

and

$$[f(x), p] \ = \ \mathrm{i}\hbar f'(x) \ , \quad [x, f(p)] = \mathrm{i}\hbar f'(p) \tag{2.3.1.4}$$

> The existence of conserving quantities is provided by a symmetry of mechanical behaviour. After the quantisation, these amounts become the generators for the algebra symmetry group.

Many types of algebras can take the finite dimensional representations [47] such as the algebra of angular momentum (2.3.1.5) illustrated by 3×3 matrices (2.2.13)). The notion:

> Let $\mathfrak{L}$ to some of abstract Lie algebra, fair a set regard to commutation relations of generators $\mathcal{J}_\alpha$. If we discover *matrices* n by n $\pi[\mathcal{J}_\alpha]_{j,k}$ fulfilling the relations of algebra, then we illustrate that we got the representation of π of the algebra $\mathfrak{L}$ at n-dimensional of vector space. The representation vector space denotes the module.

At the moment we get three vector linear spaces, the initial Euclidean space [39], where the symmetry group was characterised, provides the algebra itself as the linear space, and in expansion, we use a space of representation. From the point of view of quantum mechanical system the last one is the space of states, the space of wavefunctions [48].

2.3.2 Pauli matrices

The finite dimensional representation of the algebra angular momentum will be suggested in different illustration. The Pauli matrices [51] are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3.2.1}$$

Let

$$S_j = \frac{\sigma_j}{2} \tag{2.3.2.2}$$

One can verify,

$$[S_1, S_2] = \mathrm{i}S_3 , \quad [S_2, S_3] = \mathrm{i}S_1 , \quad [S_3, S_1] = \mathrm{i}S_2 \tag{2.3.2.3}$$

The Hermitian premise within the space of two-by-two traceless matrices [62] is framed the Pauli matrices. Also, own algebra (2.3.2.3) is named $\boldsymbol{sl_2}$ where "s" means special = traceless, "l" mentions linear space and "2" determines two by two in the two-dimensional of representation. As well, our type of algebra denotes the least complex of algebra in the finite dimensional of representations, and a third notion for it is A_1.

$$\underbrace{o(3) \text{ synonim } \boldsymbol{sl_2} \text{ synonim } A_1}_{\text{algebra of angular momentum}} \tag{2.3.2.4}$$

Precarious point is that the two-dimensional representation does not represent the group $SO(3)$. In the event that we suppose the group of component through interesting two-by-two representation,

$$u = \mathrm{e}^{\mathrm{i}\psi S_1}\mathrm{e}^{-\mathrm{i}\theta S_2}\mathrm{e}^{\mathrm{i}\phi S_3} \tag{2.3.2.5}$$

then, examine its features. The two-by-two complex-valued matrix u satisfies

$$u^\dagger u = \mathbb{1} \ , \quad \det(u) = 1 \tag{2.3.2.6}$$

Equations (2.3.2.6), very similar to (2.2.7), are the terms knowing the $SU(2)$ group, Special ($\det u = 1$) group of the Unitary two-by-two of matrices. The surprising contrast between tree-dimensional representation (2.2.11) and two-dimensional representation (2.3.2.5) is that

$$\mathrm{e}^{2\pi \mathrm{i} T_j} = \mathbb{1} \quad \text{but} \quad \mathrm{e}^{2\pi \mathrm{i} S_j} = -\mathbb{1} \tag{2.3.2.7}$$

We as a rule overlook this negative sign saying that the two-dimensional representation compares to a fermion, an odd particle with odd properties.

2.3.3 Nature of the Casimir operator

The following question is regarding to what the Casimir operator means [13]. The angular of momentum in quantum mechanical system shows up like the vector. Our duty to sense an overwhelming enticement to confirm, and we need to know about its square:

$$\mathbf{L}^2 \;=\; L_1^2 + L_2^2 + L_3^3 \tag{2.3.3.1}$$

Check utilizing the following $[AB, C] = A[B, C] + [A, C]B$ and the relation of algebra at L_j,

$$[\mathbf{L}^2, L_j] = 0 \tag{2.3.3.2}$$

One may see, $\mathbf{L}^2$ for the representations (2.2.13) and (2.3.2.1) is a multiple of unity matrix,

$$\mathbf{T}^2 = 2\ , \quad \mathbf{S}^2 = \frac{3}{4} \tag{2.3.3.3}$$

Quadratic operator (2.3.3.1), replacing at all elements of algebra, is the Casimir operator. The finite dimensional of representation is styled irreducible when the Casimir operator is a multiple of the identity matrix.

2.3.4 Basics of quantum mechanics

Our closest point is to distinguish the finite dimensional of irreducible representations for sl_2. Our matrix style is not exceptionally helpful into our objective. We maybe select other sort of bases in the module representation. An alter of the matrices, $A \to A' = SAS^{-1}$, gives likeness change of premise $\boldsymbol{e} \to \boldsymbol{e}' = S\mathbf{e}$. Without a doubt, the relations of algebra can be basis-independent:

$$A' = SAS^{-1}\ , \quad B' = SBS^{-1} \quad \Rightarrow \quad [A', B'] = SABS^{-1} - SBAS^{-1} = S[A, B]S^{-1} \tag{2.3.4.1}$$

The purpose of the basis-independent dialect within the essence of quantum mechanical system is useful. The operator acting at n-dimensional space is considered by A, and some basis is supposed by $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$. Behaviour of operator A is realised at this basis as

$$A \cdot |v_j\rangle \;=\; \sum_k |v_k\rangle \; A_{k,j} \tag{2.3.4.2}$$

Moreover, the matrix of elements for the operator A at this specific basis are $A_{k,j}$ $|v_1\rangle, \ldots, |v_n\rangle$. However, the conjugated basis is specified by $\langle v_k|v_j\rangle = \delta_{j,k}$; so,

$$A_{k,j} = \langle v_k|A|v_j\rangle \tag{2.3.4.3}$$

2.3.5 Some of quantum theories

We specify recently, the space of states is pertaining to the representation of space in quantum mechanical area. Commonly, in the framework of quantum fields, the quantum theories, also equals to the finite-dimensional of representation symmetry group [52].

Let quantum framework comprises at the scope of fields $\boldsymbol{\phi}_a$, as well the indices are represented by $\mathcal{I} \ni a$. There exist the unitary operator $\mathbf{U}(u)$ performing the symmetry group of transformation

$$\mathbf{U}(u)\boldsymbol{\phi}_a\mathbf{U}(u)^{-1} \;=\; \sum_{b\in\mathcal{I}} \boldsymbol{\phi}_a M_{ba}(u) \; . \tag{2.3.5.1}$$

Here u infers to the element for the symmetry group basic of representation such as the vector representation (2.2.11) for the rotation group. The universal symmetry of generator can be defined like that

$$\mathbf{U}(\mathbb{1} + \varepsilon T_\alpha) \;=\; \mathbb{1} + \varepsilon \mathcal{J}_\alpha \tag{2.3.5.2}$$

The representation symmetry group is $\boldsymbol{\phi}_a$, for that

$$M_{ba}(\mathbb{1} + \varepsilon T_\alpha) = \delta_{ba} + \varepsilon\pi[\mathcal{J}_\alpha]_{ba} \tag{2.3.5.3}$$

where the "$\alpha \in \mathcal{I}$"representation of the symmetry algebra element $\mathcal{J}_\alpha$ is $\pi[\mathcal{J}_\alpha]_{ba}$. The algebra relation produces the following equation

$$[\mathcal{J}_\alpha, \boldsymbol{\phi}_a] \;=\; \sum_{b\in\mathcal{I}} \boldsymbol{\phi}_b \pi[\mathcal{J}_\alpha]_{ba} \tag{2.3.5.4}$$

The vectors and scalars that shape the quantum-mechanics [53] is defined by the algebra of angular momentum. A noticeable C is named the scalar when

$$[L_j, C] = 0 \tag{2.3.5.5}$$

For example, the following $\mathbf{x}^2$, $\mathbf{p}^2$ and $\mathbf{L}^2$ are defined as scalars. A set of onservables A_1, A_2, A_3 is called the vector when

$$[L_i, A_j] \ = \ \mathsf{i} \sum_k \epsilon_{ijk} A_k \tag{2.3.5.6}$$

Triplet L_i itself is the vector.

2.4 Representations of $\boldsymbol{sl_2}$

To describe representations of $\boldsymbol{sl_2}$, it is convenient to change its basis.
Instead of generators T_1, T_2, T_3 let us define $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ by means of

$$\boldsymbol{e} = T_1 + \mathsf{i}T_2, \quad \boldsymbol{f} = T_1 - \mathsf{i}T_2, \quad \boldsymbol{h} = 2T_3 \tag{2.4.1}$$

They satisfy

$$[\boldsymbol{e}, \boldsymbol{f}] = \boldsymbol{h}, \quad [\boldsymbol{h}, \boldsymbol{e}] = 2\boldsymbol{e}, \quad [\boldsymbol{h}, \boldsymbol{f}] = 2\boldsymbol{f} \tag{2.4.2}$$

The finite-dimensional representation of $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ is

$$\mathbb{V}_j = \begin{cases} \boldsymbol{e}|j, m\,\rangle = |j, m+1\,\rangle \sqrt{(j-m)(j+m+1)} \\ \boldsymbol{f}|j, m\,\rangle = |j, m-1\,\rangle \sqrt{(j+m)(j-m+1)} \\ \boldsymbol{h}|j, m\,\rangle = |j, m\,\rangle \cdot 2m \end{cases} \tag{2.4.3}$$

where

$$j,\ m \in \mathbb{Z}/2 \quad \text{and} \quad -j \leq m \leq j$$

This $(2j{+}1)$ - dimensional representation is called the representation

with the highest weight j,

$$\boldsymbol{e}|j, j\,\rangle = 0 \tag{2.4.4}$$

and the lowest weight -j,

$$\boldsymbol{f}|j,-j\,\rangle = 0 \tag{2.4.5}$$

In physics, this representation is also called " Spin j - representation". Note that in general the "lowest" and the "highest" weights are the minimal and maximal eigenvalues of $\boldsymbol{h}$ (or of $\boldsymbol{h}/2$).

Note that for the finite-dimensional representations T_1, T_2, and T_3 are Hermitian, so that

$$\boldsymbol{e}^\dagger = \boldsymbol{f} \tag{2.4.6}$$

The infinite-dimensional representations usually are defined by a lowest weight only. In particular, we will use the infinite-dimensional representation

$$\begin{cases} \boldsymbol{e}|n\,\rangle = |n+1\,\rangle \mathrm{i}(n+1) \\ \boldsymbol{f}|n\,\rangle = |n-1\,\rangle \mathrm{i}n \\ \boldsymbol{h}|n\,\rangle = |n\,\rangle(2n+1) \end{cases} \tag{2.4.7}$$

where

$$n \in \mathbb{Z}_{n\geq 0}$$

The representation above has the lowest weight 1, i.e., it is " Spin $-1/2$ - representation".

More detailed representation theory of $\boldsymbol{sl_2}$, and in particular its relation to the representation of quantum oscillator will be discussed in Section 5.

2.5 Simple Lie algebras

2.5.1 Some common features of Lie algebras

A Lie algebra is defined by the structure constants as,

$$[\mathcal{J}_\alpha, \mathcal{J}_\beta] = \sum_\gamma c_{\alpha\beta;\gamma}\mathcal{J}_\gamma \tag{2.5.1.1}$$

The anti-symmetry of the commutator here implies $c_{\alpha\beta;\gamma} = -c_{\beta\alpha;\gamma}$. Also, the commutator provides the Jacobi identity,

$$[\mathcal{J}_\alpha, [\mathcal{J}_\beta, \mathcal{J}_\gamma]] + [\mathcal{J}_\beta, [\mathcal{J}_\gamma, \mathcal{J}_\alpha]] + [\mathcal{J}_\gamma, [\mathcal{J}_\alpha, \mathcal{J}_\beta]] = 0 \qquad (2.5.1.2)$$

what produces the Bianci identity [54] for the structure constants,

$$\sum_\mu c_{\beta\gamma;\mu} c_{\alpha\mu;\nu} + c_{\gamma\alpha;\mu} c_{\beta\mu;\nu} + c_{\alpha\beta;\mu} c_{\gamma\mu;\nu} = 0 \ . \qquad (2.5.1.3)$$

The Bianci identity means that we can't guess easily a form of an "abstract" Lie algebra. The good idea is to begin from some specific one (least complex, principal) of representation and identify legitimate structure constants.

The definition's abstract algebra (2.5.1.1) is known by its structure constants. Any kind of such finite dimensional algebra gets the very least one representation,

$$\pi_{\mathrm{ad}}[\mathcal{J}_\alpha]_{\gamma\beta} \ = \ c_{\alpha\beta;\gamma} \qquad (2.5.1.4)$$

generally called the adjoint representation The algebra connection (2.5.1.1) for the adjoint representation (2.5.1.4) is precisely the Bianci identity (2.5.1.3). We were really close to the adjoint of representation during the subsection 2.3.5, bestow condition (2.3.5.4) and the algebra definition (2.5.1.1):

$$[\mathcal{J}_\alpha, \underbrace{\mathcal{J}_\beta}_{\phi_\beta}] = \sum_\gamma c_{\alpha\beta;\gamma} \mathcal{J}_\gamma \equiv \sum_\gamma \underbrace{\mathcal{J}_\gamma}_{\phi_\gamma} \pi_{\mathrm{ad}}[\mathcal{J}_\alpha]_{\gamma\beta} \qquad (2.5.1.5)$$

For the adjoint representation of representation space that is the algebra itself.

What is "*simple Lie algebra*": let $\mathcal{J}_\alpha$ be some element of $\mathfrak{L}$. Consider the collection of

$$\mathcal{J}_\alpha, \quad [\mathfrak{L}, \mathcal{J}_\alpha] \ , \quad [\mathfrak{L}, [\mathfrak{L}, \mathcal{J}_\alpha]] \ , \quad \text{etc.} \qquad (2.5.1.6)$$

Then the Lie algebra is simple if the whole $\mathfrak{L}$ is listed above for any initial $\mathcal{J}_\alpha$. For instance, $\mathfrak{L}$ that equals $\boldsymbol{sl_2}$ makes $\mathcal{J}_1$. So

$$\mathcal{J}_1 \ , \quad [\mathfrak{L}, \mathcal{J}_1] = \mathcal{J}_2 \text{ or } \mathcal{J}_3 \quad \Rightarrow \quad \text{whole algebra is reproduced.} \qquad (2.5.1.7)$$

Counter example: The x, p and $\mathbb{1}$ are part of the Heisenberg algebra $\mathfrak{D}$. Take its element x, at that point

$$x \ , \quad [\mathfrak{D}, x] = \mathbb{1} \ , \quad [\mathfrak{D}, [\mathfrak{D}, x]] = 0 \ , \quad \text{momentum disappears.} \tag{2.5.1.8}$$

For that, we deduce that the Heisenberg algebra is not simple. Heisenberg algebra is central.

2.5.2 Algebras $\boldsymbol{sl_n}$

Our definition of n is $n = 3$, common sufficient. In the fundamental representation of Algebra $\boldsymbol{sl_3}$, that is the algebra for 3 by 3 traceless matrices. The style of matrix units is very helpful. Suppose

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{etc.} \tag{2.5.2.1}$$

The basis of the linear space of traceless 3 by 3 matrices can be like this

$$E_{ij} \ , \quad i \neq j \quad \text{and} \quad H_1 = E_{11} - E_{12} \ , \quad H_2 = E_{22} - E_{33} \tag{2.5.2.2}$$

For the matrix units, if we utilize the commutation law, then

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{ki} \tag{2.5.2.3}$$

one can effectively confirm that these eight matrices characterise basic Lie algebra. According the equation (2.5.2.3), that implies the structure constants about $\boldsymbol{sl_3}$, also for algebra $\boldsymbol{sl_n}$when $i, j, k, l = 1, ..., n$).

In mathematics, (2.5.2.2) is the foremost helpful for mathematicians, but for physicists, other bases can be utilized. The matrices H_1 and H_2 are Hermitian, one can chose the Hermitian basis when E_{ij}, $i \neq j$,

$$S_{ij} = E_{ij} + E_{ji} \ , \quad A_{ij} = \mathsf{i}(E_{ij} - E_{ji}) \ , \quad i < j \tag{2.5.2.4}$$

Algebra sl_n with the basis (2.5.2.4) is gotten (special unitary group $SU(n)$)

$$u^{\dagger}u = \mathbb{1} \ , \quad \det(u) = 1 \tag{2.5.2.5}$$

Let the neighborhood of unity such that,

$$u \ = \ \mathbb{1} \ + \ \mathrm{i}\varepsilon M \tag{2.5.2.6}$$

with real ε, and from $\det(\mathbb{1} + \varepsilon M) = 1 + \varepsilon \mathrm{Trace} M + O(\varepsilon^2)$ we come to

$$M^{\dagger} = M \ , \quad \mathrm{Trace} M = 0 \ , \tag{2.5.2.7}$$

the same terms of (2.5.2.4).

2.5.3 Cartan-Weyl basis

Turn the commutation relations (2.5.2.3) for the algebra $\boldsymbol{sl_3}$. Introduce

$$\begin{array}{l} E_1 = E_{12} \ , \quad F_1 = E_{21} \ , \quad H_1 = E_{11} - E_{22} \\ E_2 = E_{23} \ , \quad F_2 = F_{32} \ , \quad H_2 = E_{22} - E_{33} \\ - \\ E_{1+2} = E_{13} = [E_1, E_2] \ , \quad F_{1+2} = E_{31} = [F_2, F_1] \end{array} \tag{2.5.3.1}$$

The first two lines give the Cartan-Weyl basis for sl_3 [55]. So far we use the term "basis" as a minimal set of algebra generators which commutators produce the whole algebra.

The baisis of diagonal traceless matrices is represented by the elements H_1 and H_2. Any matrix commuting with H_1 and H_2 must got structure aH_1+bH_2. A Cartan subalgebra $\mathfrak{H}$ is for any Lie algebra, the maximal set of mutually commuting celements. Currently, the basis of Cartan subalgebra $\mathfrak{H}$ [56] is H_1 and H_2.

Additionally, within any simple Lie algebra of the Cartan-Weyl basis of the number of H-s rises to the number for E-s equates to the number for F-s, where that number is named rank of the algebra. The commutation relations among H and E, among H and F, and also among E and F of the Cartan-Weyl basis can be like this:

$$[E_i, F_j] = \delta_{i,j} H_i \ , \quad [H_i, E_j] = A_{i,j} E_j \ , \quad [H_i, F_j] = -A_{i,j} F_j \tag{2.5.3.2}$$

where, in the sl_3 situation, that is matrix $A_{i,j}$ is

$$||A_{i,j}|| = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.5.3.3}$$

For any matrix $A_{i,j}$ and Lie algebra, showing up in (2.5.3.2), is named the Cartan matrix, where the normalised Cartan matrices [57] are used. So,

$$A_{j,j} = 2 \tag{2.5.3.4}$$

it continuously re-scaling E_j and F_j can be performed.
The elements E_{1+2} and F_{1+2} can be checked and confirm

$$[E_{1+2}, F_{1+2}] = H_1 + H_2\ , \quad [H_i, E_{1+2}] = (A_{i,1} + A_{i,2})E_{1+2} \tag{2.5.3.5}$$

The relations can be presumed specifically from the identification of E_{1+2} and F_{1+2} (2.5.3.1) with no saying of the 3 by 3 representation.

In mathematics we find the Cartan-Weyl theorem: Any simple Lie algebra can be identified within the Cartan-Weyl basis $\{E_i, F_i, H_i\}$ by the relations (2.5.3.2) with a normalised (2.5.3.4) Cartan matrix $A_{i,j}$ [58]. The non-positive integers are off-diagonal matrix elements of Cartan matrix correspondingly. Our commutators are the elements E_i or F_i; however, these commutators must terminate somewhere. Due to algebra is finite, these commuting must void somewhere, particularly

$$\underbrace{[E_i, [E_i, \cdots [E_i}_{1-A_{i,j}\ \text{times}}, E_j] \cdots]] = 0 \tag{2.5.3.6}$$

and analogously for F. Relations (2.5.3.6) are called the Serre relations [60].

The Simple Lie algebra distinctly can be characterised via its own Cartan matrix. There exists seven classes of legal Cartan matrices [59]:

- – Cartan matrices of $\boldsymbol{sl_{r+1}}$ is A_r
- – Cartan matrices of $o(2r+1)$ is B_r
- – Cartan matrices of $sp(2r)$ is C_r: the algebra for symplectic group $SP(2r)$ conserving the antisymmetry of symplectic form of $2r$ the dimensional phase space

- – Cartan matrix for $o(2r)$ is D_r

and, as exceptional Cartan matrices:

- F_4
- G_2
- E_6, E_7, E_8

The rank of the algebras is the record all over.
The Lie algebras and their representations of entire theory are based on the single trivial observation.The normalisation for the Cartan matrix should be as

$$[E_j, F_j] = H_j \ , \quad [H_j, E_j] = 2E_j \ , \quad [H_j, F_j] = -2F_j \tag{2.5.3.7}$$

It implies, each triplet E_j, F_j, H_j is just $\boldsymbol{sl_2}$ subalgebra of the entire algebra. We know everything about the algebra $\boldsymbol{sl_2}$. For any type of the *finite-dimensional* representation of straightforward the Lie algebra can be built like a set for *finite-dimensional* submodules of all its subalgebras $\boldsymbol{sl_2}$ subalgebras.

2.6 Simple Lie algebras and their representation theory

The simple Lie algebras with the representation theory [2] [50] utilizing the perception concerning $\boldsymbol{sl_2}$ subalgebras will be our interest for this section.

2.6.1 Algebra general instance

The main action is the same as $\boldsymbol{sl_2}$ sample . The maximal commutative subalgebra is the Cartan subalgebra [17] [56], we can choose the basis of Cartan elements of eigenvectors at any representation:

$$H_i|v_{\boldsymbol{\lambda}}\rangle \ = \ |v_{\boldsymbol{\lambda}}\rangle\lambda_i \tag{2.6.1.1}$$

The weight for the state $|v_{\boldsymbol{\lambda}}\rangle$ is the column vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r)$ for eigenvalues of H_i.

What E_j and F_j do is our second action:

$$H_i E_j |v_{\boldsymbol{\lambda}}\rangle \; = \; (E_j H_i + A_{i,j} E_j)|v_{\boldsymbol{\lambda}}\rangle = E_j |v_{\boldsymbol{\lambda}}\rangle(\lambda_i + A_{i,j}). \tag{2.6.1.2}$$

Here, $A_{i,j}$ denotes the Cartan matrix. Application of E_j, by doing this, changes the vector weights. Suppose that the vector

$$\boldsymbol{\alpha}_j = (A_{i,j})_{i=1,\ldots,r} \tag{2.6.1.3}$$

is the Cartan matrix of j th column, where that is named the root. At that point (2.6.1.2) merely implies

$$E_j |v_{\boldsymbol{\lambda}}\rangle \;\sim\; |v_{\boldsymbol{\lambda}+\boldsymbol{\alpha}_j}\rangle \;, \quad F_j |v_{\boldsymbol{\lambda}}\rangle \;\sim\; |v_{\boldsymbol{\lambda}-\boldsymbol{\alpha}_j}\rangle \tag{2.6.1.4}$$

Note, $(\boldsymbol{\alpha}_j)_j = A_{j,j} = 2$, hence

$$H_j |v_{\boldsymbol{\lambda}\pm\boldsymbol{\alpha}_j}\rangle \;=\; |v_{\boldsymbol{\lambda}\pm\boldsymbol{\alpha}_j}\rangle(\lambda_j \pm 2) \tag{2.6.1.5}$$

Typically the exact same ± 2 as in the representation theory concept of $\boldsymbol{sl_2}$. Thus the E_j, F_j as well as H_j replicates exactly the sl_2 representation at j th element of the weight of vector $\boldsymbol{\lambda}$. We might move on (2.6.1.4):

$$E_j |v_{\boldsymbol{\lambda}}\rangle \;=\; c_{\lambda_j}\; |v_{\boldsymbol{\lambda}+\boldsymbol{\alpha}}\rangle \;, \quad F_j |v_{\boldsymbol{\lambda}}\rangle \;=\; d_{\lambda_j}\; |v_{\boldsymbol{\lambda}-\boldsymbol{\alpha}}\rangle \tag{2.6.1.6}$$

On the off opportunity that we are looking at the finite dimensional of representation in the Lie algebra of space V [20][21][42][43]. Any weight of vector $|v_{\boldsymbol{\lambda}}\rangle$ needs to belong to the finite dimensional for representations of all basis triplets E_j, F_j, H_j. Hence, all elements λ_j have to be integers and should frame Λ, $\Lambda - 2$, ..., $-\Lambda$ series.

There should exist a state whose weight $\boldsymbol{\lambda}$ perhaps a most elevated weight for all E_j, F_j, H_j at the same time for any kind of the finite dimensional of representation. It perhaps occurs at the off possibility that all $\lambda_j \geq 0$, like as the weights are named dominant weights. The most basic dominating weights such $\binom{1}{0}$ or $\binom{0}{1}$ are named basic weights. A state $|v_{\boldsymbol{\lambda}}\rangle$ through frustrating weight $\boldsymbol{\lambda}$ is a highest possible weight of state if

$$E_j \, |v_{\boldsymbol{\lambda}}\rangle \;=\; 0 \quad \text{for all } j \tag{2.6.1.7}$$

2.6.2 Algebra $\boldsymbol{sl_3}$

We can define the Cartan matrix for $\boldsymbol{sl_3}$ by $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Its columns give us the roots, $\boldsymbol{\alpha}_1 = \binom{2}{-1}$ and $\boldsymbol{\alpha}_2 = \binom{-1}{2}$.

Suppose that the representation through the highest of weight $\boldsymbol{\lambda}_1 = \binom{1}{0}$ is basic weight. Hence,

$$H_1|v_{\binom{1}{0}}\rangle = |v_{\binom{1}{0}}\rangle \;, \quad H_2|v_{\binom{1}{0}}\rangle = 0 \tag{2.6.2.1}$$

represents the one of most noteworthy weight of state produces two-dimensional representation of E_1, F_1, H_1 (sl_2 weight is 1) and also the trivial of representation of E_2, F_2, H_2 (sl_2 weight is 0). After that

$$F_1|v_{\binom{1}{0}}\rangle \;=\; |v_{\binom{1}{0}-\boldsymbol{\alpha}_1}\rangle \;=\; |v_{\binom{-1}{1}}\rangle \;, \quad F_2|v_{\binom{1}{0}}\rangle \;=\; 0 \tag{2.6.2.2}$$

With the weight $\binom{-1}{1}$, if the second state for a hypothetical representation shows up, then

$$H_1|v_{\binom{-1}{1}}\rangle = -|v_{\binom{-1}{1}}\rangle \;, \quad H_2|v_{\binom{-1}{1}}\rangle = |v_{\binom{-1}{1}}\rangle \tag{2.6.2.3}$$

As prospective, the lowest one for $(EFH)_1$ is the second state, however it appears to become the most elevated for $(EFH)_2$. Directly,

$$F_1|v_{\binom{-1}{1}}\rangle \;=\; 0 \;, \quad F_2|v_{\binom{-1}{1}}\rangle \;=\; |v_{\binom{-1}{1}-\boldsymbol{\alpha}_2}\rangle \;=\; |v_{\binom{0}{-1}}\rangle \tag{2.6.2.4}$$

The most reduced one is the third state because the elements of its weight are non-positive and also F_1 and F_2 must annihilate it. We have created the three-dimensional irreducible representation for $\boldsymbol{sl_2} = A_2$. Signifying $|v_{\boldsymbol{\lambda}_1}\rangle = |v_1\rangle$, $|v_{\boldsymbol{\lambda}_1-\boldsymbol{\alpha}_1}\rangle = |v_2\rangle$ as well as $|v_{\boldsymbol{\lambda}_1-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}\rangle = |v_3\rangle$, and compiling (2.6.2.1 - 2.6.2.4), we might iterate the matrix components of Cartan-Weyl generators:

$$\begin{aligned} \pi(F_1) &= E_{21}\ , \quad \pi(H_1) = E_{11} - E_{22}\ , \quad \pi(E_1) = E_{12} \\ \pi(F_2) &= E_{32}\ , \quad \pi(H_2) = E_{22} - E_{33}\ , \quad \pi(E_2) = E_{23} \end{aligned} \tag{2.6.2.5}$$

entrusted by (2.5.3.1): we reproduce the fundamental representation.

Similarly, we construct the representation through highest weight $\boldsymbol{\lambda}_2 = \binom{0}{1}$. Then, we get the representation

$$\begin{aligned} \pi(F_1) &= E_{32}\ , \quad \pi(H_1) = E_{22} - E_{33}\ , \quad \pi(E_1) = E_{23} \\ \pi(F_2) &= E_{21}\ , \quad \pi(H_2) = E_{11} - E_{22}\ , \quad \pi(E_2) = E_{12} \end{aligned} \tag{2.6.2.6}$$

which is the "conjugated" fundamental representation.

There exists two states with zero weight $\binom{0}{0}$ inside the module $V_{\binom{1}{1}}$. It is since $F_1F_2|v_{\binom{1}{1}}\rangle \neq F_2F_1|v_{\binom{1}{1}}\rangle$. At that point $\dim V_{\binom{1}{1}} = 8$. This module is precisely the adjoint of representation, the two zero of weight states equal to two elements of Cartan subalgebra, as well as the weights $\binom{2}{-1}$ and $\binom{-1}{2}$ are precisely the roots.

Ready to presently use the ideas of expansion for angular momentum to the algebra $\boldsymbol{sl_3}$ representations. We need to know how to decompose the following $V_{\binom{1}{0}} \otimes V_{\binom{1}{0}}$ for the irreducible representations, then the answer is to observe at the highest weight of vectors through the tensor product. Clearly, that first one is entirety for the highest weights, $\binom{2}{0}$, also most elevated weight shall establish the six-dimensional of module $V_{\binom{2}{0}}$. However there exist one more weight from the class of overwhelming weights. Therefore, $\binom{0}{1}$ at this module

$$V_{\binom{1}{0}} \otimes V_{\binom{1}{0}} = V_{\binom{2}{1}} \oplus V_{\binom{0}{1}}\ , \quad 3 \times 3 = 6 + \overline{3} \tag{2.6.2.7}$$

In the same time,

$$V_{\binom{1}{0}} \otimes V_{\binom{0}{1}} = V_{\binom{1}{1}} + V_{\binom{0}{0}}\ , \quad 3 \times \overline{3} = 8 + 1\ . \tag{2.6.2.8}$$

We find at this illustration, the reason for the off-diagonal components of the Cartan matrix should be non-positive integers. Other hand, there would certainly not become the finite dimensional of representations at all.

Everything works, this can be the legitimate Cartan matrix for simple Lie algebra. To be specific, $B_2 \equiv C_2$, describing either $o(5)$ (five-dimensional vector representation) or $sp(4)$ (four-dimensional vector representation). The four-dimensional module for $o(5)$ is the spinor representation. Ten-dimensional adjoint representation is $V_{\binom{2}{0}}$.

Chapter 3

On eigenstates for some sl_2 related Hamiltonian

This chapter is published as [63].

3.1 Introduction

This paper [63] deals with the representation theory of the classical Lie algebra [2]. We will consider the Lie algebra $\boldsymbol{sl_2}$ in a certain infinitely dimensional representation corresponding to the lowest weight 1. The representation module is equivalent to the Fock Space representation of the quantum oscillator [3]. The "creating" and "annihilation" operators $\boldsymbol{e}$ and $\boldsymbol{f}$ are anti-unitary, so that the operator $\boldsymbol{H} = \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$ is Hermitian, and therefore it can be interpreted as a Hamiltonian for a certain quantum mechanical system. This Hamiltonian is related to a Hamiltonian considered in [4, 5] in the limit $q = 1$ (Note, the regime $q = 1$ was not considered in [4, 5]).

This Chapter is organised as follows. In Section 2 we fix the proper representation of $\boldsymbol{sl_2}$ and rewrite the stationary Schrödinger equation as a linear recursion with non-constant coefficients. Section 3 is devoted to the analysis of the recursion equations. Its asymptotic is discussed in Section 4. Section 5 contains discussion and conclusion.

3.2 Formulation of the problem

We consider the algebra $\boldsymbol{sl_2}$ generated by three operators $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ satisfying the three fundamental commutation relations [2].

$$[\boldsymbol{e}, \boldsymbol{f}] \;=\; \boldsymbol{h} \;, \quad [\boldsymbol{h}, \boldsymbol{e}] \;=\; 2\boldsymbol{e} \;, \quad [\boldsymbol{h}, \boldsymbol{f}] \;=\; -2\boldsymbol{f} \tag{3.2.1}$$

Let F stand for the Fock space,

$$F \;=\; \mathrm{Span}\Big\{ \,|\,n\rangle \;, \quad n \in \mathbb{Z}_{n\geq 0} \,\Big\} \tag{3.2.2}$$

The map

$$\boldsymbol{e} \;\stackrel{\pi}{\to}\; \pi(\boldsymbol{e}) \;\in\; \mathrm{End}\,(F) \;, \quad \text{etc.}, \tag{3.2.3}$$

we define as

$$\boldsymbol{e}\,|n\rangle \;=\; |n{+}1\rangle\; \mathsf{i}(n{+}1) \;, \quad \boldsymbol{f}\,|n\rangle \;=\; |n{-}1\rangle\; \mathsf{i}n \;, \quad \boldsymbol{h}\,|n\rangle \;=\; |n\rangle\;(2n{+}1) \;, \quad n \in \mathbb{Z}_{n\geq 0} \;, \tag{3.2.4}$$

where for shortness we use notation $\boldsymbol{e}$ instead of $\pi(\boldsymbol{e})$, etc. Our representation (3.2.4) is the representation with the lowest weight 1,

$$\boldsymbol{h}|0\rangle \;=\; |0\rangle \tag{3.2.5}$$

(in Physics this is called "spin $= -1/2$ representation"). The Fock co-module is defined by

$$\langle n|n'\rangle \;=\; \delta_{n,n'} \;, \quad n, n' \geq 0 \tag{3.2.6}$$

An essential feature of our Chapter is that this representation not unitary:

$$\boldsymbol{e}^{\dagger} \;=\; -\boldsymbol{f} \tag{3.2.7}$$

where the "dagger" means the Hermitian conjugation. Subject of our interest is self-conjugated unbounded Hamiltonian

$$\boldsymbol{\mathcal{H}} \;=\; \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}} \tag{3.2.8}$$

and the stationary Schrödinger equation for it,

$$\boldsymbol{\mathcal{H}}\;|\psi\rangle \;=\; |\psi\rangle\; E \tag{3.2.9}$$

In what follows, we will study the structure of $|\psi\rangle$ for any $E \in \mathbb{R}$ and deduce that our Hamiltonian has continuous spectrum.

3.3 Analysis of the recursion

We will use the Dirac notations for $\langle\text{bra}|$ and $|\text{ket}\rangle$ vectors [24]. In components,

$$\psi_n \;=\; \langle n|\psi\rangle \tag{3.3.1}$$

where $\langle n|$ is a state of Fock co-module, cf. (3.2.6), and $|\psi\rangle$ is a required wavefunction.

The stationary Schrödinger equation (3.2.9) in components reads

$$(n+1)\;\psi_{n+1} \;+\; n\;\psi_{n-1} \;=\; E\;\psi_n \tag{3.3.2}$$

where we assume

$$\psi_0 \;=\; 1 \qquad \forall\;\; E \;\in\; \mathbb{R} \tag{3.3.3}$$

Our aim now is to understand the asymptotic behaviour of ψ_n when $n \to \infty$. Since E for now is only one free parameter, we assume implicitly

$$|\psi\rangle \;=\; |\psi_E\rangle\;, \quad \psi_n \;=\; \psi_n(E) \tag{3.3.4}$$

Recursion (3.3.2) can be identically rewritten in matrix form [4, 5]:

$$(\psi_n, \psi_{n+1}) \;=\; (\psi_{n-1}, \psi_n) \;\cdot\; L_{n+1} \tag{3.3.5}$$

where

$$L_n \;=\; \begin{pmatrix} 0 & -1+\dfrac{1}{n} \\ 1 & \dfrac{E}{n} \end{pmatrix} \tag{3.3.6}$$

Thus,

$$(\psi_{n-1}, \psi_n) \;=\; (0,1)\; L_1 \cdot L_2 \cdots L_{n-1} \cdot L_n \tag{3.3.7}$$

Since

$$L_\infty \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;, \quad L_\infty^4 \;=\; 1 \tag{3.3.8}$$

we expect *mod* 4 pattern for ψ_n. Diagonalising matrix L_n ,

$$L_n \;=\; P_n^{-1} \begin{pmatrix} \lambda_n & 0 \\ 0 & \overline{\lambda_n} \end{pmatrix} P_n \tag{3.3.9}$$

where

$$\lambda_n \;=\; \mathsf{i}\left(\sqrt{1-\frac{1}{n}-\frac{E^2}{4n^2}} - \mathsf{i}\frac{E}{2n}\right) \;=\; \mathsf{i}\sqrt{1-\frac{1}{n}}\;\exp\left\{-\mathsf{i}\arcsin\frac{E}{2\sqrt{n(n-1)}}\right\} \tag{3.3.10}$$

and

$$P_n P_{n+1}^{-1} \;=\; 1 + \frac{1}{2n^2}\begin{pmatrix} 0 & 0 \\ -E & 1 \end{pmatrix} + \mathcal{O}(1/n^3) \tag{3.3.11}$$

one can deduce the following asymptotic identity straightforwardly from (3.3.7):

$$\psi_n(E) \;=\; \frac{A_n(E)}{\sqrt{n}}\,\cos\left(\frac{E}{2}\log n - \frac{\pi n}{2} + \varphi_n(E)\right)\,, \quad n \geq 1 \tag{3.3.12}$$

Intensive numerical computations allow one to conclude that the sequences $A_n(E)$ and $\varphi_n(E)$ smoothly converge to $A(E)$ and $\varphi(E)$ when $n \to \infty$. Therefore, we can postulate the $1/n$ expansion for A_n and φ_n:

$$A_n(E) \;=\; A(E)\,(1 + \frac{\delta_1}{n} + \frac{\delta_2}{n^2} + \cdots)\,, \quad \varphi_n(E) \;=\; \varphi(E) + \frac{\epsilon_1}{n} + \frac{\epsilon_2}{n^2} + \cdots \tag{3.3.13}$$

with some n-independent coefficients

$$\delta_j \;=\; \delta_j(E)\,, \quad \epsilon_j = \epsilon_j(E)\,, \quad j \;\geq\; 1 \tag{3.3.14}$$

Values of $\delta_j,\ \epsilon_j$ must follow from (3.3.2). In what follows, let us combine all correction terms in (3.3.13) into

$$\delta(n,E) \;=\; \sum_{j=1}^{\infty}\frac{\delta_j(E)}{n^j}\,, \qquad \epsilon(n,E) \;=\; \sum_{j=1}^{\infty}\frac{\epsilon_j(E)}{n^j} \tag{3.3.15}$$

To get these values, let us substitute (3.3.12) into (3.3.2). To do this in convenient way, let us introduce

$$\Phi_n \;=\; \frac{E}{n}\log_n - \frac{\pi n}{2} + \varphi_n;\,, \quad \Phi_{n+1} \;=\; \Phi_n - \frac{\pi}{2} + \alpha_n;\,, \quad \Phi_{n-1} \;=\; \Phi_n + \frac{\pi}{2} - \alpha_n' \tag{3.3.16}$$

The values of α_n and α_n' are then given by

$$\begin{aligned} \alpha_n \;&=\; \Phi_{n+1} - \Phi_n + \frac{\pi}{2} \;=\; \frac{E}{2}\log_{(n+1)} + \varphi_{n+1} - \frac{E}{2}\log_n - \varphi_n \\ &=\; \frac{E}{2}\log(1+\frac{1}{n}) + \epsilon_1(\frac{1}{n+1} - \frac{1}{n}) + \epsilon_2(\frac{1}{(n+1)^2} - \frac{1}{n^2}) + \cdots \end{aligned} \tag{3.3.17}$$

and similarly for α_n'. Let further

$$\frac{1}{n} = x \quad \Rightarrow \quad \frac{1}{n+1} = \frac{x}{1+x} = \sum_{j=1}^{\infty}(-)^{j+1}x^j \quad \text{etc.,} \tag{3.3.18}$$

so that $1/n$-expansion becomes x-expansion. Then,

$$\begin{aligned} \alpha_n \;&=\; \frac{E}{2}\log(1+x) + \epsilon_1(\frac{x}{1+x} - x) + \epsilon_2(\frac{x^2}{(1+x)^2} - x^2) + \cdots \\ &=\; \frac{E}{2}x - (\frac{E}{4} + \epsilon_1)x^2 + (\frac{E}{6} + \epsilon_1 - 2\epsilon_2)x^3 + \mathcal{O}(x^4) \end{aligned} \tag{3.3.19}$$

Value of α'_n have a similar structure.

Now we can use (3.3.16, 3.3.17 and 3.3.19) in (3.3.12 and 3.3.2):

$$\begin{aligned}\psi_n &= \frac{A_n}{\sqrt{n}}\cos(\Phi_n)\ ,\\ \psi_{n+1} &= \frac{A_{n+1}}{\sqrt{n+1}}\cos(\Phi_n-\frac{\pi}{2}+\alpha_n) = \frac{A_{n+1}}{\sqrt{n+1}}(\sin\Phi_n\cos\alpha_n+\cos\Phi_n\sin\alpha_n)\\ \psi_{n-1} &= \frac{A_{n-1}}{\sqrt{n-1}}(-\sin\Phi_n\cos\alpha'_n+\cos\Phi_n\sin\alpha'_n)\end{aligned}\tag{3.3.20}$$

Equation (3.3.2) can be written as

$$\begin{aligned}&\cos\Phi_n\left[(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\sin\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\sin\alpha'_n-E\frac{A_n}{\sqrt{n}}\right]\\ &+\sin\Phi_n\left[(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\cos\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\cos\alpha'_n-E\frac{A_n}{\sqrt{n}}\right] = 0\end{aligned}\tag{3.3.21}$$

Expressions in the square brackets are the series in $1/n$. Coefficients $\cos\Phi_n$ and $\sin\Phi_n$ are irregular. Therefore, (3.3.21) can be satisfied if and only if:

$$\begin{aligned}(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\sin\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\sin\alpha'_n-E\frac{A_n}{\sqrt{n}} &= 0\\ (n+1)\frac{A_{n+1}}{\sqrt{n+1}}\cos\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\cos\alpha'_n-E\frac{A_n}{\sqrt{n}} &= 0\end{aligned}\tag{3.3.22}$$

Each LHS of (3.3.22) is well defined series in $x = 1/n$. They must be zero, so that each coefficient in $x = 1/n$ expansion must be zero. Thus (3.3.22) provides a set of algebraic equations for δ_j, ϵ_j, where j=1,2,....

Precise form of the asymptotic corrections is the following:

$$\begin{aligned}\delta(n,E) &= -\frac{1}{4\,n}+\frac{2E^2+1}{32\,n^2}-\frac{5(2E^2-1)}{128\,n^3}+\frac{20E^4-60E^2-21}{2048\,n^4}\\ &-\frac{180E^4-1380E^2+399}{8192\,n^5}+\frac{120E^6-2540E^4+2518E^2+869}{65536\,n^6}+\mathcal{O}(n^{-7})\end{aligned}\tag{3.3.23}$$

and

$$\begin{aligned}\epsilon(n,E) &= \frac{E}{4\,n}-\frac{E(E^2-5)}{96\,n^2}+\frac{E(E^2-9)}{96\,n^3}-\frac{E(9E^4-490E^2+341)}{15360\,n^4}\\ &+\frac{E(3E^4-190E^2+375)}{2560\,n^5}-\frac{E(15E^6-2793E^4+22169E^2-7615)}{258048\,n^6}+\mathcal{O}(n^{-7})\end{aligned}\tag{3.3.24}$$

The correction terms δ_j and ϵ_j can be produced from the recursion by a bootstrap up to any order of $1/n$.

3.4 Orthogonality

There is a remarkable way to derive the inner product for two states in our model. Consider a truncated state,

$$|\psi_E^{(N)}\rangle \;=\; \sum_{n=0}^{N} |n\rangle \psi_n(E) \tag{3.4.1}$$

where $\psi_n(E)$ is defined by (3.3.2) with the initial condition $\psi_0 \;=\; 1$. Straightforward computation gives

$$\boldsymbol{\mathcal{H}}\,|\psi_E^{(N)}\rangle \;=\; |\psi_E^{(N-1)}\rangle\; E \;+\; |N\rangle\; N\psi_{N-1}(E) \;+\; |N+1\rangle\;(N+1)\psi_N(E) \tag{3.4.2}$$

Considering then

$$\langle\psi_{E'}^{(N)}|\;\boldsymbol{\mathcal{H}}\;|\psi_E^{(N)}\rangle \tag{3.4.3}$$

one deduces

$$\langle\psi_{E'}^{(N-1)}|\psi_E^{(N-1)}\rangle \;=\; \frac{N}{E-E'}\;\left(\psi_N(E)\psi_{N-1}(E') - \psi_N(E')\psi_{N-1}(E)\right) \tag{3.4.4}$$

Assuming our asymptotic for ψ_N for $N \to \infty$, one obtains

$$\langle\psi_{E'}^{(N)}|\psi_E^{(N)}\rangle \;=\; A(E')A(E)\frac{\sin\left(\dfrac{E'-E}{2}\log N + \varphi(E') - \varphi(E)\right)}{E'-E}\;, \quad N \to \infty \tag{3.4.5}$$

The limit $N \to \infty$ is well defined here. In general, this is the Fresnel integral limit [12]

$$\lim_{K\to\infty} \frac{\sin(Kx)}{x} \;=\; \pi\delta(x) \tag{3.4.6}$$

Therefore, at $N \to \infty$ one obtains

$$\langle\psi_{E'}|\psi_E\rangle \;=\; \pi A(E)^2 \delta(E-E') \tag{3.4.7}$$

In fact, this is the main result of our paper. Numerical analysis also shows that the spectrum is unbounded since

$$A(E) \;=\; A(-E) \tag{3.4.8}$$

3.5 Conclusion and discussion

In this Chapter we have considered the stationary Schrödinger equation for the self-conjugated Hamiltonian $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$, where $\boldsymbol{e}$ and $\boldsymbol{f}$ are creating and annihilation operators for the algebra $\boldsymbol{sl_2}$ considered for the infinite-dimensional representation with lowest weight equals 1, equivalent to the usual Fock Space.

The eigenvector equation for operator $\boldsymbol{\mathcal{H}}$ is the the second order recursion equation. In this Chapter we have given detailed analysis for a solution of the recursion. General expression of $\psi_n(E)$ involves four functions: $A(E)$, $\psi(E)$, $\delta_n(E)$, $\varepsilon_n(E)$, see equation (3.3.13). We give the rigorous way to define $\delta(n,E)$ and $\epsilon(n,E)$ analytically in the forms of series expansion with respect to $1/n$ and E, however the functions $A(E)$ and $\psi(E)$ are defined only numerically for real E.

The further development of the problem implies two ways: the first way is the further analysis of equation (3.3.2) in order to find analytical expressions for the asymptotic analytical functions $A(E)$ and $\varphi(E)$. The second way could be $q \neq 1$ generalisation of the problem. A preliminary analysis shows that $q \neq 1$ case leads to several unexpected mathematical phenomena.

Chapter 4

Quantum Universal Enveloping

4.1 Introduction

This scholar chapter is connected to $\mathcal{U}_q(sl_2)$ and its representations. It is defined as affiliated algebra with specific generators and relations and treated as a quantum analogue of the enveloping algebra $\mathcal{U}(sl_2)$. We show that new algebra offers two main properties with the previous sl_2 type basis and no zero divisors, and for demonstrating the issues we need to infer a few commutator formulas. We moreover consider finite dimensional representations of $\mathcal{U}$ and determine the center of $\mathcal{U}$.

4.2 Basic exchange relations for $\mathcal{U}_q(sl_2)$

Consider a fixed ground field k and an element $q \in k$ such that $q \neq 0$ and $q^2 \neq 1$. Hence, $\mathcal{U}_q(sl_2)$ represents the (associative) algebra (with 1 and over k) satisfying the following relations

$$KK^{-1} = 1 = K^{-1}K, \tag{4.2.1}$$

$$KEK^{-1} = q^2E, \tag{4.2.2}$$

$$KFK^{-1} = q^{-2}F, \tag{4.2.3}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \tag{4.2.4}$$

where E, F, K, K^{-1} are the generators.

One has at that point to require that $q^4 \neq 1$. This adaptation leads to an inadequate comparable, but occasionally more complicated theory[1].

[1]Often people use $q^2 \to q$.

The algebra $\mathcal{U}_q(sl_2)$ introduced is gathered to represent a quantum analogue of the enveloping algebra $\mathcal{U}(sl_2)$. In this Chapter, the main objective in is to show that the new algebra offers two fundamental properties with the old one: It has (3.2.1) sort premise and it has no zero divisors . In order to prove these results we have to infer several commutator formulas.

4.3 Simple involutions

Let us abbreviate $\mathcal{U}= \mathcal{U}_q(sl_2)$.

Proposition 4.1 [64].
a) *"There is a unique automorphism ω of $\mathcal{U}$ with $\omega(E) = F$, $\omega(F) = E$ and $\omega(K) = K^{-1}$. It satisfies $\omega^2 = 1$".*
b) *"There is a unique antiautomorphism τ of $\mathcal{U}$ with $\tau(E) = E$, $\tau(F) = F$ and $\tau(K) = K^{-1}$. It satisfies $\tau^2 = 1$".*

Proof.
a) Let us check if

$$(\omega(E), \omega(F), \omega(K), \omega(K^{-1})) = (F, E, K^{-1}, K) \tag{4.3.1}$$

fulfill the relations (4.2.1)-(4.2.4). For case, to see that the pictures beneath ω fulfill (4.2.2), we ought to know that $K^{-1}FK = q^{-2}F$; In any case, that takes after from (4.2.3) and (4.2.1). The other relations are checked in a comparable way. The uniqueness is apparent, since E, F, K, K^{-1} produce $\mathcal{U}$. All these generators are settled beneath ω^2, so clearly $\omega^2 = 1$.

b) We should check that

$$(\tau(E), \tau(F), \tau(K), \tau(K^{-1}) = (E, F, K^{-1}, K) \tag{4.3.2}$$

satisfies the relations (4.2.1)-(4.2.4) within the algebra $\mathcal{U}^{opp}$. Here $\mathcal{U}^{opp}$ is the algebra contradicted to $\mathcal{U}$ as a vector space. Note that the product $a \cdot b$ in $\mathcal{U}_q(sl_2)^{opp}$ is break even to the product $b \cdot a$ in $\boldsymbol{\mathcal{U}}$. So to check that the pictures beneath τ satisfy (4.2.4), we note that $E \cdot F$ - $F \cdot E = FE$ - $EF = (K^{-1}$ - $K)/(q$ - $q^{-1})$. The contentions for the other relations are comparative. The uniqueness is once more self-evident; so is the equation $\tau^2 = 1$.

4.4 Basis of quantum algebra

Proposition 4.2.
"The algebra $\boldsymbol{\mathcal{U}}$ is spanned as a vector space over k by all monomials $F^s K^n E^r$ with $r,\ s,\ n \in \mathbb{Z},\ r,\ s \geq 0$."

Proof.
We claim to begin with that the span of these vectors is steady beneath multiplication by all generators of $\boldsymbol{\mathcal{U}}$ ($E,\ F,\ K,$ *and* K^{-1}). Usually trifling for F (where $FF^s K^n E^r = F^{s+1} K^n E^r$ and a simple consequence of (4.2.3) for K and K^{-1}:

$$KF^s K^n E^r = q^{-2s} F^s K^{n+1} E^r \tag{4.4.1}$$

and

$$K^{-1} F^s K^n E^r = q^{2s} F^s K^{n-1} E^r \tag{4.4.2}$$

Finally,

$$EF^s = F^s E + [s] F^{s-1} [K; 1-s] \ , \tag{4.4.3}$$

where the q-numbers notations are introduced,

$$[s] \ = \ \frac{q^s - q^{-s}}{q - q^{-1}} \ , \quad [K, 1-s] \ = \ \frac{Kq^{1-s} - K^{-1} q^{s-1}}{q - q^{-1}} \ , \tag{4.4.4}$$

and

$$[b+c][K;a] = [b][K;a+c] + [c][K;a-b] \tag{4.4.5}$$

for all $a,\ b,\ c \in \mathbb{Z}$.

yield

$$EF^s K^n E^r = F^s E K^n E^r + [s] F^{s-1} [K; 1-s] K^n E^r \tag{4.4.6}$$

$$= q^{-2n} F^s K^n E^{r+1} + [s] F^{s-1} [K; 1-s] K^n E^r \tag{4.4.7}$$

where we drop the final term for s = 0. Since we are able compose $[K\ ;\ 1\ \text{-}\ s]K^n$ as a polynomial in K and K^{-1}, we get the claim too in this case.

Now the claim suggests that the span of our monomials is stable beneath increase with any component in $\boldsymbol{\mathcal{U}}$, consequently contains $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}} \cdot 1$.

4.5 Highest and lowest states

Proposition 4.3.
Let M be a finite dimensional $\boldsymbol{\mathcal{U}}$-module, and q is not a root of unity. There exist integers $r, s > 0$ with $E^r M = 0$ and $F^s M = 0$.

Proof.
Denote by $k[X]$ the polynomial ring over k, where X is indeterminate.
Set

$$M_{(f)} = \{|m\rangle \in M\ :\ f(K)^n|m\rangle = 0\ for\ all\ n \gg 0\} \tag{4.5.1}$$

where $f \in k[X]$ is any irreducible polynomial.
Accordingly, we can deduce that the direct sum of the distinct $M_{(f)}$ is M. Consider the two polynomials f and g with $M_{(f)} \neq 0$ and $M_{(g)} \neq 0$, hence $M_{(f)} = M_{(g)}$ is true under the condition that f and g differ by a nonzero factor. We have $M_{(X)} = 0$, since the action of K on M is invertible.

Let $f \in k[X]$ be irreducible with $M_{(f)} \neq \emptyset$. For each $i \in \mathbf{Z}$ set f_i equal to the polynomial $f_i(X) = f(q^i X)$. This polynomial is once more irreducible, since it is the image of f beneath the one of a kind automorphism of $k[X]$ with $X \mapsto q^i X$. We know that f$(K)E = Ef(q^2 K)$ and thus inductively $f(K)E^r = E^r f(q^{2r} K) = E^r f_{2r}(K)$. This implies that $E^r M_{(f)} \subset M_{(f-2r)}$ for all $r \geq 0$.

We need to show that $M_{(f_{-2r})} = 0$ for some $r > 0$. (This will then imply $E^r M_{(f)} = 0$; since f was arbitrary, the claim takes after.) Well, assume that $M_{(f_{-2r})} \neq 0$ for all $r > 0$. Since the sum of the distinct $M_{(g)}$ is direct and since M is finite dimensional, there ought to be integers $s > r$ with $M_{(f_{2r})} = M_{(f_{2s})}$. Then f_{2r} and f_{2s} have to be corresponding. Since they have the same constant term (nonzero, since $f \neq X$), they have to be equal. Be that as it may, on the off chance f has degree n, at that point the driving coefficients differ by the factor $q^{2(s-r)n}$, which is not equal to 1, since q is not a root of unity, which yields to a contradiction.

The proof for F is similar.

4.6 Cartan subspaces - I

Let M be a $\boldsymbol{\mathcal{U}}$-module, then consider that for all $\lambda \in k$, $\lambda \neq 0$

$$M_\lambda = \{|m\rangle \in M \mid K|m\rangle = \lambda|m\rangle\} \tag{4.6.1}$$

i.e., M_λ is the eigenspace of K acting on M for the eigenvalue λ. (It is sufficient to consider $\lambda \neq 0$, since K has an inverse in $\boldsymbol{\mathcal{U}}$.) We denote by M_λ to be a weight space of M; the λ satisfying $M_\lambda \neq 0$ represent the weights of M. The sum of M_λ is direct. The relations (4.2.2) and (4.2.3) imply for all λ

$$EM_\lambda \subset M_{q^2\lambda} \quad and \quad FM_\lambda \subset M_{q^{-2}\lambda} \tag{4.6.2}$$

This shows that the sum of the M_λ represents a submodule of M. More accurately, the sum of $M_{q^{2n}\lambda}$ with $n \in \mathbb{Z}$ (for any λ) is a submodule. In case M is simple and in the event $M_\lambda \neq 0$, at that point $M = \bigoplus_n M_{q^{2n}\lambda}$. (assuming that q is not a root of unity, yields n to run over all integers, otherwise over an appropriate finite subset.) In common, there require not be any nonzero M_λ. But in the event k is algebraically closed. Moreover, if M is finite dimensional, then K incorporates a nonzero eigenspace on M, hence there exists a λ such that $M_\lambda \neq 0$.

4.7 Cartan subspaces - II

Proposition 4.4.

Suppose that q is not a root of unity. Consider that M is a finite dimensional $\boldsymbol{\mathcal{U}}$-module, and q is not a root of unity. Therefore, M represents the direct sum of its weight spaces. The form of the weights of M is: $\pm\, q^a$ with $a \in \mathbb{Z}$.

Proof.

If and only if the minimal polynomial of a finite dimentional vector space divides into linear factors, each acquiring multiplicity 1, the endomorphism of that vector space is diagonalizable . The eigenvalues are then the roots of the minimal polynomial. So we get to show that the shape of the minimal polynomial (within the uncertain X) of K acting on M is: $\prod_i (X - \lambda_i)$, where λ_i are distinct elements of the form $\pm q^a$ with a $\in \mathbb{Z}$.
There is an integer $s > 0$ such that $F^s M = 0$. Set

$$h_r = \prod_{j=-(r-1)}^{r-1} [K;\ r - s + j] \quad for\ all\ integers\ r \geq 0. \tag{4.7.1}$$

In particular, h_0 is the empty product equalt to 1. One checks presently by induction on r for $0 \leq r \leq s$ that $F^{s-r} h_r M = 0$. For $r = 0$ this holds by our choice of s. The induction step may be a longish calculation expelled to the reference section. We get for $r = s$

$$0 = h_s M = \left(\prod_{j=-(s-1)}^{s-1} (q - q^{-1})^{-1} q^j K^{-1} (K^2 - q^{-2j}) \right) M \tag{4.7.2}$$

We can drop the nonzero constant factors and apply a suitable power of K; this yields

$$0 = \left(\prod_{j=-(s-1)}^{s-1} (K^2 - q^{-2j}) \right) M = \left(\prod_{j=-(s-1)}^{s-1} (K - q^{-j})(K + q^{-j}) \right) M \tag{4.7.3}$$

Then the minimal polynomial of K acting on M divides $\prod_{j=-(s-1)}^{s-1} (K - q^{-j})(K + q^{-j})$, subsequently has the required form. (Note that $- q^a \neq + q^b$ for all a, b $\in \mathbb{Z}$: Otherwise $q^{b-a} = -1$ and $q^{2(b-a)} = 1$ contradicting the assumption of q that it is not a root of unity.)

Remark 4.1.

When having char$(k) = 2$, the proof shows that the minimal polynomial of K parts into linear factors and that all weights of M have the frame q^a with a $\in \mathbb{Z}$. On the off chance that M is straightforward, at that point this infers that it represents the direct sum of its weight spaces. However, building examples where M is not basic is much more simpler.

Take $M = k^2$ where E and F act like 0 and where K and K^{-1} both act as $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$.

4.8 Structure of the module

For each $\lambda \in k$, $\lambda \neq 0$ there is an (infinite dimensional) $\mathcal{U}$-module $M(\lambda)$ with basis $|m_0\rangle$, $|m_1\rangle$, $|m_2\rangle$, ... such that for all i

$$\begin{aligned} K\,|m_i\rangle &= \lambda q^{-2i}\,|m_i\rangle, \\ F\,|m_i\rangle &= |m_{i+1}\rangle, \\ E\,|m_i\rangle &= \begin{cases} 0, & \text{if } i = 0, \\ [i][\lambda, 1-i]\,|m_{i-1}\rangle, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.8.1}$$

In arrange to develop this module one can check that the endomorphisms of $M(\lambda)$ characterized by (4.8.1) fulfill the relations (4.2.1)-(4.2.4). Be that as it may, it is less demanding to watch that we are able to take

$$M(\lambda) = \boldsymbol{\mathcal{U}}/(\boldsymbol{\mathcal{U}}E + \boldsymbol{\mathcal{U}}(K-\lambda)) \tag{4.8.2}$$

with $|m_i\rangle$ equal to the coset of F^i. Then the formulas in (4.8.1) take after effortlessly from (4.2.3) and (4.4.3). We get the linear independence of the $|m_i\rangle$.

The portrayal in (4.8.2) makes it clear that $M(\lambda)$ has the taking after widespread property: In the event that M may be a $\boldsymbol{\mathcal{U}}$-module and $|m\rangle \in M$ a vector with $E|m\rangle = 0$ and $K|m\rangle = \lambda|m\rangle$, at that point, a unique homomorphism of $\boldsymbol{\mathcal{U}}$-modules φ takes place such that: $M(\lambda) \to M$ with $\varphi(|m_0\rangle) = |m\rangle$.

If λ is equal to $\pm q^a$ for some $a \in \mathbb{Z}$, then we can simplify the last equation in (4.8.1) as follows:

$$E\,|m_i\rangle = \pm\,[i][a+1-i]\;|m_{i-1}\rangle \qquad \textit{in case}\;\; \lambda = \pm q^a \qquad (4.8.3)$$

By (4.8.1), each $|m_i\rangle$ is contained in $M(\lambda)_{q^{-2i}\lambda}$, $q^{-2i}\lambda$ are distinct if k is not a root of unity, and we get

$$M(\lambda)_{q^{-2i}\lambda} = k\,|m_i\rangle \qquad \textit{for all}\;\; i \;\geq\; 0 \qquad (4.8.4)$$

If we have q as a primitive l-th root of unity,then we get $q^{-2i}\lambda$, if and only if, l divides 2(j-i). So, if l is odd, then the weight spaces in $M(\lambda)$ are the

$$M(\lambda)_{q^{-2i}\lambda} = \bigoplus_{n\geq 0} k\,|m_{i+nl}\rangle \qquad \textit{with}\;\; 0 \leq i \leq l \qquad (4.8.5)$$

If l = 2l' is even, then the weight spaces are the

$$M(\lambda)_{q^{-2i}\lambda} = \bigoplus_{n\geq 0} k\,|m_{i+nl'}\rangle \qquad \textit{with}\;\; 0 \leq i \leq l' \qquad (4.8.6)$$

4.9 Simple modules

Proposition 4.5.
Take the assumption of q that it is not a root of unity, and consider that $\lambda \in k$, $\lambda \neq 0$. *The* $\boldsymbol{\mathcal{U}}$*-module* $M(\lambda)$ is simple in the case of $\lambda \neq \pm \mathrm{q}^n$ for all integers ≥ 0. However, when $\lambda = \pm q^n$ *for some integer* $n \geq 0$, the m_i *with* $i \geq n+1$ *span a submodule of* M (λ) *isomorphic to* $M(q^{-2(n+1)}\lambda)$; *this submodule of* M (λ) *is the only one to be different from 0 and* $M(\lambda)$.

Proof.
Consider that M' is any nonzero submodule of M(λ). Because M' is K -stable, it represents the direct sum of its weight spaces, i.e., of all *M'* $\cap$ $\mathrm{M}(\lambda)_\mu$. Now (4.8.4) infers that *M'* is traversed by the $|m_i\rangle$ contained in *M'*. Since we accept *M'* $\neq 0$, there exists an i with $|m_i\rangle \in$ *M'*. Select $j \geq 0$ minimal with $|m_j\rangle \in$ *M'*. We have at that point $|m_i\rangle = F^{i-j}\,|m_j\rangle \in$ *M'* for all $i \geq j$, then *M'* is the span of all $|m_i\rangle$ with

$i \geq j$. if $j = 0$, then $M' = M(\lambda)$. So let us pretend $j > 0$. Because $E\,|m_j\rangle \in M'$ is a multiple of $|m_{j-1}\rangle \notin M'$ we have $|Em_j\rangle = 0$, hence λq^{1-j} - $\lambda^{-1} q^{j-1} = 0$. This implies $\lambda^2 = q^{2(j-1)}$, i.e., $\lambda = \pm q^{(j-1)}$.

The contention so far shows that $M(\lambda)$ is simple when $\lambda \neq \pm \mathrm{q}^n$ (for all $n \in \mathbf{Z}$, ≥ 0), though in case $\lambda = \pm q^n$ there is at most one submodule of $M(\lambda)$ distinctive from 0 and $M(\lambda)$. In any case, if $\lambda = \pm q^n$, at that point (4.8.3) suggests $E|m_{n+1}\rangle = 0$. So there is (by the universal property) a homomorphism $M(q^{-2(n+1)}\lambda) = M(\pm q^{-n-2}) \to M(\lambda)$ that takes the $|m_0\rangle$ in $M(\pm q^{-n-2})$ to the $|m_{n+1}\rangle$ in $M(\lambda)$. Since $M(\pm q^{-n-2})$ is basic (by what we have as of now demonstrated), usually an isomorphism onto its picture. This implies the claim within the proposition.

4.10 Parity

Proposition 4.6.

Take the consideration of q is not a root of unity. There exists *a simple $\mathcal{U}$-module $L(n, +)$* for each integer n ≥ 0 *of basis* $|m_0\rangle, |m_1\rangle, \ldots, |m_n\rangle$ and *another simple $\mathcal{U}$-module $L(n, -)$ of basis* $|m_0'\rangle, |m_1'\rangle, \ldots, |m_n'\rangle$ *for all i* where $(0 \leq i \leq n)$

$$K\,|m_i\rangle = q^{n-2i}\,|m_i\rangle, \qquad K\,|m_i'\rangle = -q^{n-2i}\,|m_i'\rangle \tag{4.10.1}$$

$$F\,|m_i\rangle = \begin{cases} |m_{i+1}\rangle, & \text{if } i < n \\ 0, & \text{if } i = n, \end{cases} \qquad F\,|m_i'\rangle = \begin{cases} |m_{i+1}'\rangle, & \text{if } i < n \\ 0, & \text{if } i = n \end{cases}$$

$$E\,|m_i\rangle = \begin{cases} [i][n+1-i]\,|m_{i-1}\rangle, & \text{if } i > 0 \\ 0, & \text{if } i = 0 \end{cases}$$

$$E\,|m_i'\rangle = \begin{cases} -[i][n+1-i]\,|m_{i-1}'\rangle, & \text{if } i > 0 \\ 0, & \text{if } i = 0 \end{cases}$$

Each simple n+1 dimension $\mathcal{U}$-module is isomorphic to $L(n, +)$ or to $L(n, -)$.

Proof.

We get the presence from proposition 4.5.: Set $L(n, \pm) = M(\pm q^n)/M'$ where M' is the submodule spanned by the $|m_i\rangle$ with $i > n$. Take for the $|m_i\rangle$ resp. $|m_i'\rangle$ the pictures of the $|m_i\rangle \in M(\pm q^n)$. The formulas in 2.4 abdicate the formulas above. The effortlessness of $L(n, \pm)$ takes after from proposition 4.5.

Let M be a simple $\mathcal{U}$-module of a finite dimension. Let us prove that M is isomorphic to a few $L(n, \pm)$. We have $M = \bigoplus_\lambda M_\lambda$ by proposition 4.3. (Utilize the remark 4.1. in 4.7 on the off chance char$(k) = 2$.) Since dim $M < \infty$, the set of λ with $M_\lambda \neq 0$ is finite. So ready to discover λ with $M_\lambda \neq 0$ and $M_{q^2\lambda} = 0$. Pick $|m\rangle \in M_\lambda$, $|m\rangle \neq 0$. We have $E|m\rangle \in M_{q^2\lambda}$ by (4.7.2), thus $E|m\rangle$=0. The widespread property implies *that there is a nonzero homomorphism* $\varphi : M_\lambda \to M$. Since M is basic, φ is surjective. Since M is finite dimensional, proposition 4.5. infers that λ=$\pm q^n$ for some $n \in \mathbb{Z}$, $n \geq 0$. Then φ induces an isomorphism between $L(n, \pm)$ and M.

Remark 4.2.

In the event k has characteristic 2, then clearly $L(n, +)$ and $L(n, -)$ are isomorphic (for each n). On the off chance that the characteristic of k is not break even with to 2, then these modules are not isomorphic: The subspace $k|m_0\rangle$ resp. $k|m_0'\rangle$ is determined as the set of all $|m\rangle$ with $E|m\rangle = 0$, and $\pm q^n$ is determined as the eigenvalue of K on this subspace.

4.11 Casimir operator

We set

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} \tag{4.11.1}$$

Using (4.2.4) we can rewrite

$$C = EF - \frac{K - K^{-1}}{q - q^{-1}} + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} \tag{4.11.2}$$

$$= EF + \frac{Kq + K^{-1}q^{-1} - (q - q^{-1})(K - K^{-1})}{(q - q^{-1})^2} \tag{4.11.3}$$

hence

$$C = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2} \tag{4.11.4}$$

We are able to express (4.11.4) moreover utilizing the maps ω and τ from 4.2 as the maps *omega* and *tau* from 4.3 as

$$\omega(C) = C = \tau(C) \tag{4.11.5}$$

Proposition 4.7.
a) *The element C is central in $\boldsymbol{\mathcal{U}}$.*
b) On each M(λ), C applies a multiplication by the scalar: $(\lambda q + \lambda^{-1}q^{-1})/(q - q^{-1})^2$.
c) *acts on both M(λ) and M(μ) by the same scalar only in the case of $\lambda = \mu$ or $\lambda = \mu^{-1}q^{-2}$.*

Proof.
a) Obviously, C is homogeneous of degree 0. So it implies that C commutes with K and K^{-1}. A rudimentary calculation shows that $EC = CE$. If we apply ω to this equation, we get $FC = CF$. So C commutes with all generators of $\boldsymbol{\mathcal{U}}$, thus it is central.
b) It is evident by (4.8.1) that $Cm_0 = \mu m_0$ where μ is the scalar in our claim. Since C commutes with F and $m_i = F^i m_0$ for all i, we get too $Cm_i = \mu m_i$ for all i, thus the claim.
c) We get by b) the same scalar if and only if $\lambda q + \lambda^{-1}q^{-1} = \mu q + \mu^{-1}q^{-1}$.

$$(\lambda - \mu)q = (\mu^{-1} - \lambda^{-1})q^{-1} = \lambda^{-1}\mu^{-1}(\lambda - \mu)q^{-1} \tag{4.11.6}$$

i.e., to $\lambda - \mu = 0$ or $q = \lambda^{-1}\mu^{-1}q^{-1}$. The claim takes after.

4.12 Uniqueness of Casimir eigenvalue

Of course C works too on each homomorphic picture of $M(\lambda)$ as scalar multiplication (by the same scalar as on $M(\lambda)$). This applies in specific to the modules presented in proposition 4.8 (if q not a root of unity).

Proposition 4.8.
Take the consideration of q is not a root of unity. Consider that L and $L^{'}$ are finite dimensional simple $\boldsymbol{\mathcal{U}}$-modules. If C applies the same scalar on both L and $L^{'}$, then L is isomorphic to $L^{'}$.

Proof.
There are integers n, $m \geq 0$ and signs ε, ε' such that L is a factor module of $M(\varepsilon q^n)$ and $L^{'}$ one of $M(\varepsilon' q^m)$. If C multiplies both L and $L^{'}$ by the same scalar, at that point too on $M(\varepsilon q^n)$ and $M(\varepsilon' q^m)$. So proposition 4.8 infers that $\varepsilon q^n = \varepsilon' q^m$ (and thus $L \simeq L^{'}$ as required) or that $\varepsilon q^n = \varepsilon' q^{-m-2}$. The second case leads to $q^{n+m+2} = \varepsilon\varepsilon' = \pm 1$ what contradicts with the requirement that q is not a root of unity.

4.13 Semi-simple modules

Proposition 4.9.
Let us suppose that q does not represent a root of unity. Consider that M is a $\mathcal{U}$-module that is finite dimensional and represents the direct sum of its corresponding weight spaces. Then we can call M as a semisimple $\mathcal{U}$-module.

Proof.
Consider that M is a $\mathcal{U}$-module that is finite dimensional. Select a composition series of M such that $= M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$. Each M_i/M_{i-1} is isomorphic to one of the modules described in 4.11, so $\boldsymbol{\mathcal{C}}$ acts by a scalar, say μ_i, on M_i/M_{i-1}. Then $\prod_{i=1}^{r}(\boldsymbol{\mathcal{C}} - \mu_i)$ destroys M. So the minimal polynomial of $\boldsymbol{\mathcal{C}}$ acting on M splits into linear factors, and M is the coordinate entirety of the generalized eigenspaces for $\boldsymbol{\mathcal{C}}$:

$$M = \bigoplus_{\mu} M_{(\mu)} \qquad Where \qquad M_{(\mu)} = \{|m\rangle \in M \mid (C-\mu)^s|m\rangle = 0 \quad for \quad s \gg 0\}. \tag{4.13.1}$$

Since $\mathcal{C}$ is central in $\mathcal{U}$, each $M_{(\mu)}$ is a submodule of M. So it is enough to prove that each $M_{(\mu)}$ is semisimple.

Let us expect that $M = M_{(\mu)}$ for a few μ. At that point $\mathcal{C}$ - μ acts nilpotently on M, subsequently on each M_i/M_{i-1}. However, $\mathcal{C}$ causes an increase by μ_i on M_i/M_{i-1}; this suggests that $\mu_i = \mu$ for all i. Presently proposition 4.6 implies that there is an integer $n \geq 0$ and a sign ε such that each M_i/M_{i-1} is isomorphic to $L(n, \varepsilon)$.

We suppose that M represents the direct sum of its corresponding weight spaces, $M = \bigoplus_v M_v$. If N is a submodule of M, then we can say that $N = \bigoplus_v N_v$ and $N_v = N \cap M_v$ for all v. This infers that dim M_v = dim N_v + dim$(M/N)_v$. In case we apply this repeteadly to the composition series, we get

$$\text{dim } M_v = \sum_{i=1}^{r} \text{dim}(M_i/M_{i-1})_v = r\, dim\; L(n, \varepsilon)_v \tag{4.13.2}$$

In specific, we get presently from 4.11 that dim $M_\lambda = r$ where $\lambda = \varepsilon q^n$ and that $M_{q^2\lambda} = 0$. For any $|v\rangle \in M_\lambda$ we have hence $E|v\rangle = 0$, so the submodule $\mathcal{U}v$ is a homomorphic picture of M_λ; since it is finite dimensional, it is isomorphic to $L(n, \varepsilon)$ (if $|v\rangle \neq 0$). Select a basis $|v_1\rangle$, $|v_2\rangle$, ... , $|v_r\rangle$ of M_λ. So, we can write M as $M = \sum_{i=1}^{r} \mathcal{U}|v_i\rangle$, since $(M/\sum_{i=1}^{r} U\,|v_i\rangle)_\lambda = 0$ and since each L that is the composition factor of $M/\sum_{i=1}^{r} U\,|v_i\rangle$ is isomorphic to $L(n, \varepsilon)$, subsequently fulfills $L_\lambda \neq 0$. By comparing measurements (dim $M = r$ dim $L(n, \varepsilon) = \sum_{i=1}^{r}$ dim $\mathcal{U}|v_i\rangle$) we see that M is in reality the direct sum of the $\mathcal{U}|v_i\rangle$, hence semisimple.

4.14 Basic statements for q-root of unity

Now, we will study the case when considering that q is a root of unity. If $q^l = 1$ then $[l] = 0$, hence $[i]^! = 0$ whenever $i \geq l > 2$.

Proposition 4.10.

If we consider q to be a primitive l-th root of unity (such that $l \in \mathbb{Z}$, $l \geq 3$), *then* E^l, F^l, K^l and K^{-l} are contained in the center of $\boldsymbol{\mathcal{U}}$.

Proof.

From (4.2.2) and (4.2.3)

$$K^lEK^{-1} = q^{2l}E = E \qquad \textit{and} \qquad K^lFK^{-l} = q^{-2l}F = F \tag{4.14.1}$$

so K^l and K^{-l} are central. We get also

$$KE^lK^{-1} = q^{2l}E^l = E^l \qquad \textit{and} \qquad KF^lK^{-1} = q^{-2l}F^l = F^l \tag{4.14.2}$$

The formulas (4.4.3) and

$$FE^r = E^rF - [r]E^{r-1}[K; r-1] \tag{4.14.3}$$

infer that $EF^l = F^lE$ and $FE^l = E^lF$ since $[l] = 0$. So too E^l and F^l are central.

Remark 4.3.

In the event l is even, $l = 2l'$, at that point as of now $[l'] = 0$ and the argument over appears that already $E^{l'}$, $F^{l'}$, $K^{l'}$, $K^{-l'}$ are central in $\boldsymbol{\mathcal{U}}$.

In the taking after subsections I select the case where q could represent a primitive *l-th* root of unity such that l is odd. The even case is comparable; more often than not one fair needs to supplant l by $l' = l/2$. Subtle elements are cleared out ...

If l is odd, assume that q is a primitive *l-th* root of unity. consider the case of $l \geq 3$, and for any b, $\lambda \in k$ with $\lambda \neq 0$ set

$$\mathbb{Z}_b(\lambda) = M(\lambda)/\boldsymbol{\mathcal{U}}(|m_l\rangle - b\,|m_0\rangle) \tag{4.14.4}$$

utilizing the notation from 4.8. We get $E|m_l\rangle = 0$ since $[l] = 0$ and $K|m_l\rangle = \lambda q^{-2l}\,|m_l\rangle = \lambda|m_l\rangle$, hence $E(|m_l\rangle - b|m_0\rangle) = 0$ and $K(|m_l\rangle - b|m_0\rangle) = \lambda(|m_l\rangle - b|m_0\rangle)$. Therefore $\boldsymbol{\mathcal{U}}(|m_l\rangle - b|m_0\rangle)$ is spanned by all $F^i(|m_l\rangle - b|m_0\rangle) = |m_{i+l}\rangle - b|m_i\rangle$ with $i \geq 0$. So the pictures within $\mathbb{Z}_b(\lambda)$ of the m_j with $j < l$ are a basis of $\mathbb{Z}_b(\lambda)$. By mishandle of notation we indicate the picture of m_j once more by $|m_j\rangle$. So the basis of $\mathbb{Z}_b(\lambda)$ is $|m_0\rangle$, $|m_1\rangle$, ... , $|m_{l-1}\rangle$, where the activity of $\boldsymbol{\mathcal{U}}$ is given by

$$K\,|m_i\rangle = q^{-2i}\lambda|m_i\rangle$$
$$F\,|m_i\rangle = \begin{cases} |m_{i+1}\rangle & \text{if } i < l-1 \\ b|m_0\rangle & \text{if } i = l-1 \end{cases}$$
$$E\,|m_i\rangle = \begin{cases} 0 & \text{if } i = 0 \\ [i][\lambda, 1-i]\,|m_{i-1}\rangle & \text{if } i > 0 \end{cases} \tag{4.14.5}$$

The $q^{-2i}\lambda$ with $0 \leq i \leq l$ are distinct because q represents a primitive l-*th* root of unity such that l odd. This implies

$$\mathbb{Z}_b(\lambda)_{q^{-2i}\lambda} = k\,|m_i\rangle \qquad for\;\; 0 \leq i \leq l \tag{4.14.6}$$

We have $F^l|m_0\rangle = b|m_0\rangle$, in reality F^l acts as increase by b on $\mathbb{Z}_b(\lambda)$. This shows that proposition 4.4 does not expand to the present case: Able to select $b \neq 0$ and after that F does not act nilpotently on the finite dimensional $\boldsymbol{\mathcal{U}}$-module $\mathbb{Z}_b(\lambda)$. We see also that K can have eigenvalues other than $\pm q^a$ (with $a \in \mathbb{Z}$) since we are able to take a λ distinct from these (unless k is finite and -q creates the multiplicative group of k). So at slightest the final part of proposition 4.6 does not generalize.

Proposition 4.11.

Consider that q is a primitive l-th root of unity such that l is odd, and $l \geq 3$. if $b \neq 0$ or if $\lambda^{2l} \neq 1$, then $\mathbb{Z}_b(\lambda)$ is a simple $\boldsymbol{\mathcal{U}}$-module. If $b = 0$ and $\lambda = \pm q^n$ with $0 \leq n \leq l$, then $\mathbb{Z}_b(\lambda)$ is considered simple only in the case of $n = l$ - 1; such that $n < l$ - 1 the m_j with $j > n$ span a submodule of $\mathbb{Z}_b(\lambda)$, and this is the only submodule different from 0 and $\mathbb{Z}_b(\lambda)$.

Proof.

We can more or less copy the proof of proposition 4.7. Let M be any nonzero submodule of $\mathbb{Z}_b(\lambda)(\lambda)$. Since M is K-stable, it is the coordinate entirety of its weight spaces, i.e., of all $M \cap \mathbb{Z}_b(\lambda)_\mu$. Now (4.14.2) infers that M is spanned by the $|m_i\rangle$ contained in M. Since we accept $M \neq 0$, there exists an i with $|m_i\rangle \in M$. Select $j \geq 0$ minimal with $|m_j\rangle \in M$. We get at that point $|m_i\rangle = F^{i-j}|m_j\rangle \in M$ for all i with $j \leq i < l$. If $j = 0$, then $M = \mathbb{Z}_b(\lambda)$. So let us suppose that $j > 0$. In the case of $b \neq 0$, $|m_0\rangle = b^{-1}F\,|m_{l-1}\rangle \in M$, hence $j = 0$. So we ought to have $b = 0$. Clearly M is the span of all $|m_i\rangle$ with $i \geq j$. Since $E\,|m_j\rangle \in M$ is a multiple of $|m_{j-1}\rangle \notin M$ we have $E\,|m_j\rangle = 0$, thus $[j](\lambda q^{1-j}$ - $\lambda^{-1}q^{j-1}) = 0$. Since $0 < j < l$ our assumption on q implies $[j] \neq 0$. We get in this manner $\lambda^2 = q^{2(j-1)}$, subsequently $\lambda^{2l} = 1$ and $\lambda = \pm q^{j-1}$.

It is left to show that for $b = 0$ and $\lambda = \pm q^n$ with $0 \leq n < l$ - 1 the span of the $|\mathrm{m}_i\rangle$ with $i > n$ is certainly a submodule. That, however, follows without problems from $E\,|m_{n+1}\rangle = 0$.

Remark 4.4.

The proposition infers that we get for $0 \leq n < l$ basic $\boldsymbol{\mathcal{U}}$-modules $L(n, +)$ and $L(n,$ -) of dimension n+1 where the formulas in proposition 4.6. portray the activity of $\boldsymbol{\mathcal{U}}$ on a reasonable premise: Fair take the special straightforward quotient of $\mathbb{Z}_0(\pm q^n)$.

The proposition implies too that the $\mathbb{Z}_0(\pm q^n)$ with $0 \leq n < l$ - 1 are not semisimple: The submodule crossed by the m_i with $i > n$ has no complement. So, proposition 4.9 does not amplify to our show circumstance.

Let q be a primitive l-th root of unity such that l odd, and $l \geq 3$. We desire to present an explanation of all simple $\boldsymbol{\mathcal{U}}$-modules M that are finite dimensional beneath the additional assumption that k is algebraically closed. This makes certain that M represents the direct sum of its corresponding weight spaces. Furthermore, usnig the way of Schur's lemma [15] the central elements E^l, F^l, K^l, and C have to act as scalars on M.

Case 1.

E^l ACTS AS 0 ON M. At that point the subspace $\{|m\rangle \in M \mid E\,|m\rangle = 0\}$ is nonzero. By (4.2.2) it is K-stable, so K has an eigenvector in it. This implies that we can discover $|m\rangle \in M$, $|\mathrm{m}\rangle \neq 0$ and $\lambda \in k$, $\lambda \neq 0$ with $E\,|m\rangle = 0$ and $K\,|m\rangle = \lambda|m\rangle$. By the universal property there exists a homomorphism $\varphi : (M\lambda) \rightarrow M$ with $\varphi(|m_0\rangle)$ $= |m\rangle$. Since M is simple, φ is surjective. A scalar $b \in k$ exists where F^l acts as multiplication by b on M. Then

$$\varphi(|m_l\rangle - b\,|m_0\rangle) = \varphi(F^l\,|m_0\rangle) - b\,|m\rangle = F^l\,|m\rangle - b\,|m\rangle = 0 \tag{4.14.7}$$

So, $\boldsymbol{\mathcal{U}}(|m_l\rangle$ - $b\,|m_0\rangle)$ is contained within the part of φ, and φ factors through $\mathbb{Z}_b(\lambda)$.

So proposition 4.7. suggests that M is either isomorphic to $\mathbb{Z}_b(\lambda)$ or to a $L(n, \pm)$.

Case 2.

F^l ACTS AS 0 ON M AND E^l DOES NOT. We use the automorphism ω from 5.2 to "twist" $\boldsymbol{\mathcal{U}}$-modules: For any $\boldsymbol{\mathcal{U}}$-module N set ${}^{\omega}N$ equal to the $\boldsymbol{\mathcal{U}}$-module that rises to N similar to a vector space such that each $u \in \boldsymbol{\mathcal{U}}$ acts on ${}^{\omega}N$ as $\omega(u)$ acts on N. It is evident that ${}^{\omega}({}^{\omega}N) \simeq N$ for all N and that ${}^{\omega}N$ is straightforward if and only if N is basic.

This infers (since ${}^{\omega}(E^l) = F^l$ and ${}^{\omega}(F^l) = E^l$) that ${}^{\omega}M$ may be a straightforward module as considered in Case 1 with $b \neq 0$. So M is isomorphic to some ${}^{\omega}\ \mathbb{Z}_b(\lambda)$. It has measurement l and there is a premise $|m_0\rangle$, $|m_1\rangle$, ... ,$|m_{l-1}\rangle$ such that the activity of $\boldsymbol{\mathcal{U}}$ is given by

$$K\,|m_i\rangle = q^{-2i}\lambda^{-1}|m_i\rangle$$
$$F\,|m_i\rangle = \begin{cases} 0, & \text{if } i = 0 \\ [i][\lambda; 1-i]\ |m_{i-1}\rangle, & \text{if } i > 0 \end{cases} \tag{4.14.8}$$
$$E\,|m_i\rangle = \begin{cases} |m_{i+1}\rangle & \text{if } i < l-1 \\ b\ |m_0\rangle & \text{if } i = l-1 \end{cases}$$

Case 3.

BOTH F^l and E^l DO NOT ACT AS 0 ON M. Let $b \in k$, $b \neq 0$ be the scalar through which F^l acts on M. We are able to discover (k is algebraically closed) and eigenvector $|m_0\rangle \neq 0$ for K in M. Indicate the comparing eigenvalue by λ. Set $|m_i\rangle$ $= F_i|m_0\rangle$ for $0 < i < l$. We have $F^{l-i}|m_i\rangle = F^l|m_0\rangle = b\,|m_0\rangle \neq 0$, hence $|m_i\rangle \neq$ 0.The activity of K and F on the $|m_i\rangle$ is given by

$$K\,|m_i\rangle = q^{-2i}\lambda\ |m_i\rangle$$
$$F\,|m_i\rangle = \begin{cases} |m_{i+1}\rangle & \text{if } i < l-1 \\ b\ |m_0\rangle & \text{if } i = l-1 \end{cases} \tag{4.14.9}$$

The $q^{-2i}\lambda$ with $0 \leq i < l$ are distinct, so the $|m_i\rangle$ are lineraly independent as eigenvectors comparing to distinct eigenvalues.

The central element C from (4.11.1) acts on M through a scalar. Since,

$$FE\,|m_0\rangle = C\,|m_0\rangle - \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}\,|m_0\rangle \tag{4.14.10}$$

there is an $a' \in k$ with $FE\,|m_0\rangle = a'\,|m_0\rangle$. We get at that point

$$bE\,|m_0\rangle = F^l E\,|m_0\rangle = F^{l-1}a'\,|m_0\rangle = a'\,|m_{l-1}\rangle \tag{4.14.11}$$

hence $E\,|m_0\rangle = a\,|m_{l-1}\rangle$ with $a = a'/b$. Now (4.4.3) yields

$$E\,|m_i\rangle = EF^i\,|m_0\rangle = F^iE\,|m_0\rangle + [i]F^{i-1}[K;1-i]\,|m_0\rangle \tag{4.14.12}$$

for all $i > 0$, hence

$$E\,|m_i\rangle = \begin{cases} a\,|m_{l-1}\rangle & \text{if } i = 0 \\ (ab + \frac{(q^i - q^{-i})(\lambda q^{1-i} - \lambda q^{-1}q^{i-1})}{(q-q^{-1})^2})\,|m_{i-1}\rangle & \text{if } i > 0 \end{cases} \tag{4.14.13}$$

This appears in specific that the span of the m_i is steady beneath all generators of $\mathcal{U}$, thus break even with to (the simple module) M. So the $|m_i\rangle$ are a basis of M, and (4.14.8) (4.14.9) depict the module totally.

On the other hand, given a, b, and λ we can use (4.14.8) (4.14.9) to characterize a $\mathcal{U}$-module. (The most thing to be checked is that the relation (4.2.4) is preserved.) On the off chance that $b \neq 0$, at that point the same argument as within the proof of 4.15 appears that the module is basic. In case a and all ab+$(q^i$ - $q^{-i})(\lambda q^{1-i}$ - $\lambda^{-1}q^{i-1})(q$ - $q^{-1})^{-2}$ are not equal to 0, then E^l does not act as 0, so we have a module of our display sort.

It ought to be famous that the module M interestingly determines b, but does not decide a and λ, since we might select rather than $|m_0\rangle$ any other $|m_i\rangle$. So we may supplant λ by $q^{-2i}\lambda$ and a by a+$(q^i$ - $q^{-i})(\lambda q^{1-i}$ - $\lambda^{-1}q^{i-1})(q$ - $q^{-1})^{-2}b^{-1}$, and still get an isomorphic module.

Remark 4.5.

For any n with $0 \leq n \leq p-1$ the module $\mathbb{Z}_0(q^n)$ is a nonsplit extension of $L(n, +)$ and $L(p - n - 2, +)$. This implies that the center of $\boldsymbol{\mathcal{U}}$ acts by the same character on these two simple modules. Additionally, it acts by the same character on $L(n, -)$ and $L(p - n - 2, -)$. Using the classification over one can check that these are the as it were cases of two straightforwaard modules that are not isomorphic, but where the center of $\boldsymbol{\mathcal{U}}$ acts by means of the same character on both.

4.15 Center of $\boldsymbol{\mathcal{U}}$

We have used within the last subsections that certain components are central in $\mathcal{U}$. We now want to determine the total center of $\boldsymbol{\mathcal{U}}$.

Review the evaluating on $\boldsymbol{\mathcal{U}}$. Indicate the graded pieces by $\boldsymbol{\mathcal{U}}_m$ with $m \in \mathbf{Z}$. By development $\boldsymbol{\mathcal{U}}_m$ is crossed by all $F^s K^n E^r$ with $m = r - s$.

Proposition 4.12.

a) $\boldsymbol{\mathcal{U}}_0$ contains the center of $\boldsymbol{\mathcal{U}}$ if q is not a root of unity.

b) *If q represents a primitive l-th root of unity such that l is odd, and $l \geq 3$, then E^l and F^l generates the center of* $\boldsymbol{\mathcal{U}}$ *, and its intersection with* $\boldsymbol{\mathcal{U}}_0$.

Proof.

The center of a grader algebra is graded. So we got to discover for each m which components in $\boldsymbol{\mathcal{U}}_m$ are central in $\boldsymbol{\mathcal{U}}$. In the event $u \in \boldsymbol{\mathcal{U}}_m$ with $u \neq 0$ is central, then it refers $q^{2m} = 1$. On the off chance where q does not represent a root of unity, then this yields $m = 0$; so a) follows. Assume presently that q reprents a primitive l-th root of unity such that l is odd, and $l \geq 3$. At this point m has to be a multiple of l. In case $m = al$ with $a > 0$, then $\mathcal{U}_m$ is traversed by all $F^s K^n E^{s+al}$. So any $u \in \boldsymbol{\mathcal{U}}_m$ can be decomposed $u = u' E^{\boldsymbol{al}}$ with $u' \in \boldsymbol{\mathcal{U}}_0$. Then u is central if and only if u' is central, since E^{al} is central and since $\boldsymbol{\mathcal{U}}$ has no zero divisors . Additionnally, if $m = -al$ with $a > 0$, then the central components in $\boldsymbol{\mathcal{U}}_m$ are precisely the products of F^{al} with central elements in $\boldsymbol{\mathcal{U}}_0$. This proves b).

Any $u \in \boldsymbol{\mathcal{U}}_0$ can be composded uniquely $u = \sum_{r \geq 0} F^r h_r E^r$ with all $h_r \in \boldsymbol{\mathcal{U}}^0$, nearly all break even with to 0.

Proposition 4.13.
let $u = \sum_{r\geq 0} F^r h_r E^r \in \boldsymbol{\mathcal{U}}_0$ *as above.Then* u *is central in* $\mathcal{U}$ *if and only if*

$$h_r - \gamma_{-2}(h_r) = [r+1][K;-r]h_{r+1} \qquad for\ \ all\ \ r \geq 0. \tag{4.15.1}$$

Proof.
We have

$$Eu = \sum_{r\geq 0} EF^r h_r E^r = \sum_{r\geq 0} F^r E h_r E^r + \sum_{r\geq 0} [r] F^{r-1}[K;1-r] h_r E^r \tag{4.15.2}$$

$$= \sum_{r\geq 0} F^r \gamma_{-2}(h_r) E^{r+1} + \sum_{r\geq 0} [r+1] F^r [K;-r] h_{r+1} E^{r+1}. \tag{4.15.3}$$

On the other hand $uE = \sum_{r\geq 0} F^r h_r E^{r+1}$, so $Eu = uE$ if and only if (4.15.1) holds. A comparable calculation shows that too $Fu = uF$ is identical to (4.15.1). At long last, $uK = Ku$ is naturally fulfilled by any $u \in \boldsymbol{\mathcal{U}}_0$.

Consider the outline $\pi : \boldsymbol{\mathcal{U}}_0 \to \boldsymbol{\mathcal{U}}^0$ that takes any $u = \sum_{r\geq 0} F^r h_r E^r$ to h_0. So π is the linear projection from $\boldsymbol{\mathcal{U}}_0$ onto the subspace $\boldsymbol{\mathcal{U}}^0$ with part $F\boldsymbol{\mathcal{U}}_0 E$. We claim that π is indeed an algebra homomorphism. It is sufficient to show that the bit could be a two-sided perfect. So take a typical premise component $F^r K^n E^r$ (with $r,n \in \mathbb{Z}$, $r > 0$) of the part and duplicate it with an self-assertive premise component $F^s K^m E^s$ (with $s,m \in \mathbb{Z}$, $s \geq 0$) of $\boldsymbol{\mathcal{U}}^0$. In the event $s > 0$, at that point clearly both $(F^s K^m E^s)(F^r K^n E^r)$ and $(F^r K^n E^r)(F^s K^m E^s)$ are in $F\boldsymbol{\mathcal{U}}_0 E$. On the off chance that $s = 0$, then $K^m(F^r K^n E^r) = q^{-rm} F^r K^{n+m} E^r = (F^r K^n E^r)K^m$, so again both products are in $F\boldsymbol{\mathcal{U}}_0 E$.

Chapter 5

Representation theory of sl_2, $\mathcal{U}_q(sl_2)$, oscillator, and q-oscillator: "physicist's approach"

This chapter is under publication as [66].

5.1 Introduction

In this Chapter we pay attention to the oscillator algebra and the Fock space, and we readdress the representation theory of sl_2 where we restate the construction of right and left module for explicit homomorphism using quantum mechanic approach. We reconsider some essential notations for the q-deformation of sl_2. We reframe the Schrödinger problem for the quantum deformed algebras, which gives the leverage of the extensives of the representation theory. Consequently, we portray shortly another class of representations of $\mathcal{U}_q(sl_2)$.

5.2 Oscillator algebra and the Fock space

Algebra sl_2 is generated by three elements e, f, h satisfying the well-known exchange relations

$$[e, f] = h \ , \quad [h, e] = 2e \ , \quad [h, f] = -2f \tag{5.2.1}$$

In previous sections, we have used an infinite-dimensional representation of $\boldsymbol{sl_2}$ with the structure of the Fock space representation of the quantum oscillator algebra.

$$\begin{cases} \boldsymbol{e}|n\rangle = |n+1\,\rangle\; \mathrm{i}(n+1) \\ \boldsymbol{f}|n\rangle = |n-1\,\rangle\; \mathrm{i}n \quad\; n \geq 0 \\ \boldsymbol{h}|n\rangle = |n\,\rangle(2n+1) \end{cases} \tag{5.2.2}$$

The quantum oscillator algebra $\mathcal{O}$ is generated by operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ called "annihilation" and "creation" operators,

$$\boldsymbol{a}\boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{a} = 1, \quad N = \boldsymbol{a}^\dagger\boldsymbol{a} \tag{5.2.3}$$

Here "dagger" stands for Hermitian conjugation and self-conjugated operator $\boldsymbol{N}$ is called "the occupation number operator". The quantum oscillator algebra admits the representation over the infinite-dimensional right module called the Fock space. Its states are denoted by

$$|n\rangle\;, n = 0, 1, 2, ... \tag{5.2.4}$$

The state $|0\rangle$ is called the Fock vacuum, annihilated by $\boldsymbol{a}$, the other states are the result of application of creation operators to the vacuum,

$$\boldsymbol{a}|0\rangle = 0\;, \quad |n\rangle = \frac{1}{\sqrt{n!}}(\boldsymbol{a}^\dagger)^n|0\rangle \tag{5.2.5}$$

The action of $\boldsymbol{a}$, $\boldsymbol{a}^\dagger$ and N are then given by

$$\begin{cases} \boldsymbol{a}|n\rangle = |n-1\rangle\sqrt{n} \\ \boldsymbol{a}^\dagger|n\rangle = |n+1\rangle\sqrt{n+1} \\ N|n\rangle = |n\rangle n \end{cases} \tag{5.2.6}$$

Here it worth to mention few convenient relations for an enveloping of the oscillator algebra,

$$\boldsymbol{a}(\boldsymbol{a}^\dagger)^n = (\boldsymbol{a}^\dagger)^n\boldsymbol{a} + n(\boldsymbol{a}^\dagger)^{n-1} \tag{5.2.7}$$

$$\boldsymbol{a}F(\boldsymbol{a}^\dagger) = F(\boldsymbol{a}^\dagger)\boldsymbol{a} + F(\boldsymbol{a}^\dagger) \tag{5.2.8}$$

Also note

$$\boldsymbol{a}F(N) = F(N+1)\boldsymbol{a} \tag{5.2.9}$$

and

$$\boldsymbol{a}^\dagger F(N) = F(N-1)\boldsymbol{a}^\dagger \tag{5.2.10}$$

Note the remarkable feature of the representation (5.2.1): Instead of

$$E^{\dagger} = F$$

for finite-dimensional representations, here one has

$$\boldsymbol{e}^{\dagger} = -\boldsymbol{f}$$

In our case, explicit homomorphism from oscillator algebra to $\boldsymbol{sl_2}$ is

$$\begin{cases} \boldsymbol{e} = \mathrm{i}\sqrt{N}\boldsymbol{a}^{\dagger} = \mathrm{i}\boldsymbol{a}^{\dagger}\sqrt{N+1} \\ \boldsymbol{f} = \mathrm{i}\sqrt{N+1}\boldsymbol{a} = \mathrm{i}\boldsymbol{a}\sqrt{N} \\ \boldsymbol{h} = 2N+1 \end{cases} \tag{5.2.11}$$

5.3 Representation theory of $\boldsymbol{sl_2}$ revisited

Here we will repeat construction of right and left module for (5.2.1) using a "quantum mechanical approach" [16].

The exchange relations (5.2.1) can be rewritten in the following form:

$$\boldsymbol{he} = \boldsymbol{e}(\boldsymbol{h}+2)\ , \quad \boldsymbol{fe} = \boldsymbol{ef} - \boldsymbol{h} \tag{5.3.1}$$

so that

$$\boldsymbol{fe}^n = \boldsymbol{e}^n\boldsymbol{f} - \boldsymbol{e}^{n-1}n(\boldsymbol{h}+n-1) \tag{5.3.2}$$

The lowest weight right vector (an analogue of the Fock vacuum) can be introduced is on usual way,

$$\boldsymbol{f}|\psi_0\ \rangle = 0 \quad \& \quad \boldsymbol{h}|\psi_0\ \rangle = |\psi_0\ \rangle\lambda_0 \tag{5.3.3}$$

where $\lambda_0 \in \mathbb{C}$ is the lowest weight of the corresponding right module.

Two remarkable relations based on (5.2.1) and (5.2.2) could be mentioned:

$$\boldsymbol{f}^n\boldsymbol{e}^n|\psi_0\ \rangle \ = \ |\psi_0\rangle\,(-1)^n\,n!\,\lambda_0(\lambda_0+1)\cdots(\lambda_0+n-1) \tag{5.3.4}$$

and

$$\boldsymbol{he}^n = \boldsymbol{e}^n(\boldsymbol{h}+2n) \tag{5.3.5}$$

Since conjugation *a priori* is not defined, one has to consider the right module and its co-module (left module) in the same time:

$$\langle\bar{\psi}_0|\psi_0\rangle \ = \ 1\ , \quad \boldsymbol{h}\,|\psi_0\rangle \ = \ |\psi_0\rangle\,\lambda_0\ , \quad \langle\bar{\psi}_0|\,\boldsymbol{h} \ = \ \lambda_0\,\langle\bar{\psi}_0| \tag{5.3.6}$$

and

$$\boldsymbol{f}\;|\psi_0\rangle \;=\; 0, \quad \langle\bar{\psi}_0|\;\boldsymbol{e} \;=\; 0 \tag{5.3.7}$$

Then, the whole right module can be defined by

$$|\psi_n\rangle \;=\; \boldsymbol{e}^n|\psi_0\rangle \tag{5.3.8}$$

and its co-module,

$$\langle\bar{\psi}_m|\psi_n\rangle \;=\; \delta_{m,n} \tag{5.3.9}$$

Such construction of $|\psi_n\rangle$, *a priori* infinite dimensional, is called a free module. We denote it as $\mathbb{M}_{\;\lambda_0}$,

$$|\psi_n\rangle \;\in\; \mathbb{M}_{\lambda_0}\;,$$

contrary to the finite dimensional module[1] M_{λ_0}. With this definition of right states, the $\boldsymbol{sl_2}$ elements act as follows:

$$\begin{cases} \boldsymbol{e}|\psi_n\;\rangle = |\psi_{n+1}\;\rangle \\ \boldsymbol{f}|\psi_n\;\rangle = |\psi_{n-1}\;\rangle\;(-)n(\lambda_0+n-1) \qquad \| \quad \boldsymbol{f}|\psi_0\;\rangle = 0 \\ \boldsymbol{h}|\psi_n\;\rangle = |\psi_n\;\rangle\;(\lambda_0+2n) \end{cases} \tag{5.3.10}$$

Then, for the left co-module, one has correspondingly

$$\begin{cases} \langle\;\bar{\psi}_n|\boldsymbol{e} = \;\langle\;\bar{\psi}_{n-1}\;| \qquad \| \quad \langle\;\bar{\psi}_0|\;\boldsymbol{e} = 0 \\ \langle\;\bar{\psi}_n|\boldsymbol{f} = -(n+1)(\lambda_0+n)\;\langle\;\bar{\psi}_{n+1}\;| \\ \langle\;\bar{\psi}_n|\boldsymbol{h} = (\lambda_0+2n)\;\langle\;\bar{\psi}_n\;| \end{cases} \tag{5.3.11}$$

The general definition of co-module states in accordance to (5.3.9) is

$$\langle\bar{\psi}_n| \;=\; \frac{(-)^n}{n!\lambda_0(\lambda_0+1).....(\lambda_0+n-1)}\;\langle\bar{\psi}_0|\;\boldsymbol{f}^n \tag{5.3.12}$$

The Casimir operator in "variables" $\boldsymbol{e},\boldsymbol{f},\boldsymbol{h}$ is defined by

$$\boldsymbol{C} \;=\; \boldsymbol{e}\boldsymbol{f} + \frac{1}{4}\,\boldsymbol{h}^2 - \frac{1}{2}\,\boldsymbol{h} \tag{5.3.13}$$

Its eigenvalue on $\mathbb{M}_{\lambda_0}$ is

$$\boldsymbol{C}\;|\psi_n\rangle \;=\; |\psi_n\rangle\;\frac{\lambda_0(\lambda_0-2)}{4}\;, \quad \forall\; n \tag{5.3.14}$$

[1] We use subscript "$-\lambda_0$" since λ_0 is the lowest weight vector, usually negative.

In what follows, the $\mathbf{\Gamma}$-function notations are useful. The $\mathbf{\Gamma}$-function is defined by

$$\mathbf{\Gamma}(z) = \int_0^\infty t^{z-1}\boldsymbol{e}^{-t}dt \tag{5.3.15}$$

Its fundamental property

$$\mathbf{\Gamma}(z) = (z-1)\mathbf{\Gamma}(z-1)\ , \quad \mathbf{\Gamma}(n) = (n-1)! \tag{5.3.16}$$

allows one to use the short notation

$$\lambda(\lambda_0+1).....(\lambda_0+n-1) = \frac{\mathbf{\Gamma}(\lambda_0+n)}{\mathbf{\Gamma}(\lambda_0)} \tag{5.3.17}$$

Equation (5.3.12) takes short form

$$\langle\ \bar{\psi}_n\ | = \frac{(-)^n\Gamma(\lambda_0)}{n!\Gamma(\lambda_0+n)}\ \langle\ \bar{\psi}_0|\boldsymbol{f}^n \tag{5.3.18}$$

The finite dimensional representations correspond to the existence of the highest vector $|\psi_N\rangle\ \in\ \mathbb{M}_{\lambda_0}$,

$$\boldsymbol{e}\ |\psi_N\rangle = 0 \qquad (i.e. \quad \boldsymbol{f}\ |\psi_{N+1}\rangle\ \equiv\ 0) \tag{5.3.19}$$

so that the lowest weight is fixed to be non-positive integer $-N$

$$\boldsymbol{f}|\psi_{N+1}\rangle\ =\ |\psi_N\rangle\ \Big(-(N+1)(\lambda_0+N)\Big)\ =\ 0 \tag{5.3.20}$$

so that

$$\lambda_0 = -N \quad \text{and} \quad 0 \le n \le N \tag{5.3.21}$$

If so,

$$\begin{cases} \boldsymbol{f}|\psi_n\rangle \ = \ |\psi_{n-1}\rangle\ n(N+1-n) \\ \boldsymbol{e}|\psi_n\rangle \ = \ |\psi_{n+1}\rangle \\ \boldsymbol{h}|\psi_n\rangle \ = \ |\psi_n\rangle\ (2n-N)\ , \quad \text{Spectrum of } \boldsymbol{h}\ =\ \{-N, -N+2, ..., N\} \end{cases} \tag{5.3.22}$$

Thus one obtains the $N+1$ dimensional representation $\mathbb{V}_N$, and

$$\mathbb{M}_N\ =\ \mathbb{V}_N\ \oplus\ \mathbb{M}_{-N-1}$$

what is exactly (2.4.3). Module $\mathbb{V}_N$ is called “spin $j = \frac{N}{2}$ representation of angular momentum” in quantum mechanics[2]., The corresponding formulas for the co-module

[2]Its dimension is $2j+1$, and the eigenvalue of the Casimir operator (5.3.13) is $j(j+1)$.

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$$\begin{cases} \langle\bar{\psi}_n|\,\boldsymbol{e} \;=\; \langle\bar{\psi}_{n-1}| \\ \langle\bar{\psi}_n|\,\boldsymbol{f} \;=\; (n+1)(N-n)\;\langle\bar{\psi}_{n+1}| \\ \langle\bar{\psi}_n|\,\boldsymbol{h} \;=\; (2n-N)\;\langle\bar{\psi}_n| \end{cases} \tag{5.3.23}$$

Standard change of normalisation for the finite dimensional representation

$$|\phi_n\rangle = |\psi_n\rangle\; C_n\;, \qquad \langle\bar{\phi}_n| = \frac{1}{C_n}\;\langle\bar{\psi}| \tag{5.3.24}$$

where

$$\begin{cases} \dfrac{C_n}{C_{n+1}} = \sqrt{(n+1)(N-n)} \\ \\ \dfrac{C_{n-1}}{C_n} = \sqrt{n(N+1-n)} \end{cases} \qquad \text{so that} \qquad C_n = C_0\sqrt{\frac{(N-n)!}{n!N!}} \tag{5.3.25}$$

provides the more convenient matrix elements,

$$\begin{cases} \boldsymbol{e}|\phi_n\rangle = \;|\phi_{n+1}\rangle\;\sqrt{(n+1)(N-n)} \\ \boldsymbol{f}|\phi_n\rangle = \;|\phi_{n-1}\rangle\;\sqrt{n(N+1-n)} \\ \langle\bar{\phi}_n|\boldsymbol{f} = \;\sqrt{(n+1)(N-n)}\;\langle\bar{\phi}_{n+1}| \\ \langle\bar{\phi}_n|\boldsymbol{e} = \;\sqrt{n(N+1-n)}\;\langle\bar{\phi}_{n-1}| \end{cases} \tag{5.3.26}$$

so that

$$\boldsymbol{f} \;=\; \boldsymbol{e}^\dagger$$

and the cut-off relations $\boldsymbol{f}|\phi_0\rangle = 0$ and $\boldsymbol{e}|\phi_N\rangle = 0$ become more natural. In compact form, the new basis is defined by

$$|\phi_n\rangle = \sqrt{\frac{(N-n)!}{N!n!}}\;\boldsymbol{e}^n|\phi_0\rangle \tag{5.3.27}$$

Now we consider our case

$$\lambda_0 = 1 \qquad (\text{spin} = -\frac{1}{2}) \tag{5.3.28}$$

The set (5.3.22) becomes

$$\begin{cases} \boldsymbol{e}\,|\psi_n\rangle \;=\; |\psi_{n+1}\rangle \\ \boldsymbol{f}\,|\psi_n\rangle \;=\; |\psi_{n-1}\rangle\;(-n^2) \qquad\qquad n \geq 0 \\ \boldsymbol{h}\,|\psi_n\rangle \;=\; |\psi_n\rangle\;(2n+1) \end{cases} \tag{5.3.29}$$

and (5.3.23) becomes

$$\begin{cases} \langle\bar{\psi}_n|\,\boldsymbol{e} \;=\; \langle\bar{\psi}_{n-1}| \\ \langle\bar{\psi}_n|\,\boldsymbol{f} \;=\; -(n+1)^2\,\langle\bar{\psi}_{n+1}| \\ \langle\bar{\psi}_n|\,\boldsymbol{h} \;=\; (2n+1)\,\langle\bar{\psi}_n| \end{cases} \tag{5.3.30}$$

As before, it is convenient to introduce another normalisation,

$$|\phi_n\rangle \;=\; |\psi_n\rangle\, C_n\;, \qquad \langle\bar{\phi}_n| \;=\; \frac{1}{C_n}\,\langle\bar{\psi}_n| \tag{5.3.31}$$

where

$$\frac{C_n}{C_{n+1}} \;=\; \mathrm{i}\,(n+1); \quad \frac{C_{n-1}}{C_n} \;=\; \mathrm{i}\,n\;, \quad \text{so that} \quad C_n \;=\; \frac{1}{\mathrm{i}^n n!}$$

In the new basis one has

$$\boldsymbol{e}\,|\phi_n\rangle \;=\; |\phi_{n+1}\rangle\,\mathrm{i}\,(n+1) \quad | \quad \langle\bar{\phi}_n|\,\boldsymbol{f} \;=\; \mathrm{i}\,(n+1)\,\langle\bar{\phi}_{n+1}|$$

$$\boldsymbol{f}\,|\phi_n\rangle \;=\; |\phi_{n-1}\rangle\,\mathrm{i}\,n \quad | \quad \langle\bar{\phi}_n|\,\boldsymbol{e} \;=\; \mathrm{i}\,n\,\langle\bar{\phi}_{n-1}| \tag{5.3.32}$$

5.4 $\mathcal{U}_q(sl_2)$ and q-oscillator revisited

In this section we will review some essential notations for the q-deformation of $\boldsymbol{sl_2}$. The quantum enveloping $\mathcal{U}_q(\boldsymbol{sl_2})$ of $\boldsymbol{sl_2}$ is defined by the following exchange relations for three elements $\boldsymbol{e}, \boldsymbol{f}$ and $\boldsymbol{k}$,

$$[\boldsymbol{e},\boldsymbol{f}] \;=\; (q-q^{-1})\,(\boldsymbol{k}-\boldsymbol{k}^{-1})\;; \quad \boldsymbol{ke}=q^2\boldsymbol{ek}\;; \quad \boldsymbol{kf}=q^{-2}\boldsymbol{fk} \tag{5.4.1}$$

This definition differs from the standard one by a renormalisation of $\boldsymbol{e}, \boldsymbol{f}$. I did it for the shortness in what follows. The element $\boldsymbol{k}$ is formally related to the Cartan element [17] of $\boldsymbol{sl_2}$ by

$$\boldsymbol{k} = q^{\boldsymbol{h}}$$

When $q \in \mathbb{C}$ is in general position, the representation theory of the quantum enveloping is essentially the same as the representation theory for underlying algebra. The free module $\mathbb{M}_{\lambda_0}$ can be defined similarly to (5.3.3) and (5.3.8),

$$\boldsymbol{f}\,|\psi_0\rangle \;=\; 0\;; \quad |\psi_n\rangle \;=\; \boldsymbol{e}^n|\psi_0\rangle\;; \quad \boldsymbol{k}|\psi_n\rangle \;=\; |\psi_n\rangle\, q^{\lambda_0+2n} \tag{5.4.2}$$

Action of the elements $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{k}$ on this basis is given then by

$$\begin{cases} \boldsymbol{e}|\psi_n\rangle &= |\psi_{n+1}\rangle \\ \boldsymbol{f}|\psi_n\rangle &= |\psi_{n-1}\rangle \left(-(q^n - q^{-n})\,(q^{\lambda_0+n-1} - q^{-\lambda_0-n+1}) \right) \\ \boldsymbol{k}|\psi_n\rangle &= |\psi_n\rangle\, q^{\lambda_0+2n} \end{cases} \tag{5.4.3}$$

The mid equation in (5.4.3) is based on

$$\boldsymbol{f}\boldsymbol{e}^n = \boldsymbol{e}^n\boldsymbol{f} - \boldsymbol{e}^{n-1}(q^n - q^{-n})(q^{n-1}\boldsymbol{k} - q^{-n+1}\boldsymbol{k}^{-1}) \tag{5.4.4}$$

The Casimir operator for $\boldsymbol{\mathcal{U}_q(sl_2)}$ is defined by

$$\boldsymbol{C} = \boldsymbol{e}\boldsymbol{f} + \boldsymbol{q^{-1}k} + \boldsymbol{qk^{-1}} \tag{5.4.5}$$

Its eigenvalue on $\mathbb{M}_{\lambda_0}$ is

$$\boldsymbol{C}|\Psi_n\rangle = |\Psi_n\rangle(q^{\lambda_0-1} + q^{1-\lambda_0}) \tag{5.4.6}$$

It its turn, the q-deformed oscillator algebra $\mathcal{O}_q$ with the generators $\boldsymbol{a}^+, \boldsymbol{a}^-$ and $\boldsymbol{k}$, is defined by the following set of exchange relations:

$$\begin{cases} \boldsymbol{a}^+\boldsymbol{a}^- = 1 - q^{-1}\boldsymbol{k} \\ \boldsymbol{a}^-\boldsymbol{a}^+ = 1 - q\boldsymbol{k} \\ \boldsymbol{k}\boldsymbol{a}^+ = q^{-2}\boldsymbol{a}^+\boldsymbol{k} \\ \boldsymbol{k}\boldsymbol{a}^- = q^2\boldsymbol{a}^-\boldsymbol{k} \end{cases} \tag{5.4.7}$$

The element $\boldsymbol{k}$ is now related to N by

$$\boldsymbol{k} \;=\; q^{2N+1}$$

This algebra allows the representation over the Fock space with

$$\boldsymbol{a}^-|0\rangle \;=\; 0\,, \quad \boldsymbol{k}\,|0\rangle \;=\; |0\rangle\, q\,, \quad |n\rangle \;=\; (\boldsymbol{a}^+)^n|0\,\rangle$$

On this basis

$$\begin{cases} \boldsymbol{a}^-|n\rangle &= |n-1\rangle\,(1 - q^{2n}) \\ \boldsymbol{a}^+|n\rangle &= |n-1\rangle \\ \boldsymbol{k}|n\rangle &= |n\rangle\, q^{2n+1} \end{cases} \tag{5.4.8}$$

However, it is convenient to change the normalization of the states as follows:

$$\begin{cases} \boldsymbol{a}^-|n\rangle = |n-1\rangle\ \sqrt{1-q^{2n}} \\ \boldsymbol{a}^+|n\rangle = |n+1\rangle\ \sqrt{1-q^{2n+2}} \\ \boldsymbol{k}|n\rangle = |n\rangle\ q^{2n+1} \end{cases} \tag{5.4.9}$$

so that

$$\left(\boldsymbol{a}^-\right)^\dagger \ = \ \boldsymbol{a}^+ \tag{5.4.10}$$

for real $q \ < \ 1$.

It is also convenient to change the normalisation of the states for the finite dimensional representation of $\boldsymbol{\mathcal{U}_q(sl_2)}$ corresponding to the choice $\lambda_0 = -N$. Let the new states $|\phi_n\rangle$ are related to the previously defined states $|\psi_n\rangle$ by

$$|\phi_n\ \rangle = |\psi_n\ \rangle\ C_n \tag{5.4.11}$$

where coefficients satisfy the recursion relation

$$\frac{C^2_{n-1}}{C^2_n} \ = \ q^{-N-1}(1-q^{2n})(1-q^{2(N+1-n)}) \tag{5.4.12}$$

The solution for C_n is then

$$C_n = \frac{q^{\frac{1}{2}n(N+1)}}{\sqrt{(q^2;q^2)_n(q^{2(N+1-n)};q^2)_n}} \tag{5.4.13}$$

where

$$(x;q^2)_n \ = \ (1-x)(1-xq^2)\cdots(1-xq^{2n-2}) \ = \ \prod_{j=0}^{n-1}(1-xq^{2j}) \tag{5.4.14}$$

is the Pochgammer symbol [18], a q-analogue of the factorial. This provides the more suitable matrix elements,

$$\begin{cases} \boldsymbol{f}|\phi_n\rangle \ = \ |\phi_{n-1}\rangle\ q^{-(N+1)/2}\sqrt{(1-q^{2n})(1-q^{2(N+1-n)})} \\ \\ \boldsymbol{e}|\phi_n\rangle \ = \ |\phi_{n+1}\rangle\ q^{-(N+1)/2}\sqrt{(1-q^{2(n+1)})(1-q^{2(N-n)})} \end{cases} \tag{5.4.15}$$

In this case, $\boldsymbol{e}$ and $\boldsymbol{f}$ are conjugated for the real q,

$$\boldsymbol{e}^\dagger = \boldsymbol{f} \tag{5.4.16}$$

Now we can look at our situation,

$$\lambda_0 = 1 \qquad (spin = -\frac{1}{2}) \tag{5.4.17}$$

The normalisation (5.4.11) now is

$$|\phi_n\rangle = |\psi_n\rangle\, C_n\ , \tag{5.4.18}$$

where

$$\frac{C_{n-1}}{C_n} = \mathrm{i}\, q^{-n}(1-q^{2n}) \tag{5.4.19}$$

and therefore

$$C_n \;=\; \frac{q^{n(n+1)/2}}{\mathrm{i}^n (q^2;q^2)_n} \tag{5.4.20}$$

The set (5.4.3) turns into ,

$$\begin{cases} \boldsymbol{e}|\phi_n\ \rangle = |\phi_{n+1}\ \rangle\ \mathrm{i}q^{-n-1}(1-q^{2(n+1)}) \\ \boldsymbol{f}|\phi_n\ \rangle = |\phi_{n-1}\ \rangle\ \mathrm{i}q^{-n}(1-q^{2n}) \\ \boldsymbol{k}|\phi_n\ \rangle = |\phi_n\ \rangle\ q^{2n+1} \end{cases} \tag{5.4.21}$$

The right co-module, defined by the standard pairing $\langle\bar{\phi}_n|\phi_n\rangle = \delta_{m,n}$, has the symmetric action of $\boldsymbol{e}, \boldsymbol{f}$:

$$\begin{cases} \langle\bar{\phi}_n|\boldsymbol{e} = \mathrm{i}q^{-n}(1-q^{2n})\ \langle\bar{\phi}_{n-1}| \\ \langle\bar{\phi}_n|\boldsymbol{f} = \mathrm{i}q^{-n-1}(1-q^{2(n+1)})\ \langle\bar{\phi}_{n+1}| \end{cases} \tag{5.4.22}$$

Note now, the representation is symmetric but anti-unitary[3],

$$\boldsymbol{e}^T \;=\; \boldsymbol{f} \qquad \text{but} \qquad \boldsymbol{e}^\dagger \;=\; -\boldsymbol{f} \tag{5.4.23}$$

Now one can establish the homomorphism from the q-oscillator algebra $\mathcal{O}_q$ and its Fock space representation (5.4.9) and $\lambda_0 = 1$ representation (5.4.17) of $\boldsymbol{\mathcal{U}_q(sl_2)}$,

$$\begin{cases} \boldsymbol{e} \;=\; \mathrm{i}\ (q^{-1}\boldsymbol{k})^{-1/2}\sqrt{1-q^{-1}\boldsymbol{k}}\ \boldsymbol{a}^+ \;=\; \mathrm{i}\,\boldsymbol{a}^+\ (q\boldsymbol{k})^{-1/2}\sqrt{1-q\boldsymbol{k}} \\ \boldsymbol{f} \;=\; \mathrm{i}\,\boldsymbol{a}^-\ (q^{-1}\boldsymbol{k})^{-1/2}\sqrt{1-q^{-1}\boldsymbol{k}} \;=\; \mathrm{i}\ (q\boldsymbol{k})^{-1/2}\sqrt{1-q\boldsymbol{k}}\ \boldsymbol{a}^- \end{cases} \tag{5.4.24}$$

while the element $\boldsymbol{k}$ is the same in both cases. In this case

$$|\phi_n\rangle \;\equiv\; |n\rangle\ , \qquad \langle\bar{\phi}_n| \;\equiv\; \langle n| \tag{5.4.25}$$

5.5 Hamiltonians in $\mathcal{U}_q(sl_2)$ case

In this section we will reformulate the Schrödinger problem for the quantum deformed algebras.

[3]Symbol “T” stands for transposition, and “†” stands for Hermitian conjugation.

One example of a self-conjugated Hamiltonian is

$$\mathcal{H} \;=\; \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}} \tag{5.5.1}$$

Its eigenvalue problem

$$\mathcal{H}\,|\Psi\rangle \;=\; |\Psi\rangle\, E \tag{5.5.2}$$

in components

$$\langle n|\frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}|\Psi\rangle \;=\; \langle n|E|\Psi\,\rangle \tag{5.5.3}$$

gives the recursion

$$q^{-n}(1-q^{2n})\Psi_{n-1} + q^{-n-1}(1-q^{2(n+1)})\;\Psi_{n+1} = E\Psi_n \tag{5.5.4}$$

where

$$\langle n|\Psi\rangle \;=\; \Psi_n \tag{5.5.5}$$

Below we present some analysis of (5.5.4).

Firstly, (5.5.4) may be rewritten in the form of the matrix multiplication,

$$(\Psi_{n+1},\Psi_n) = (\Psi_n,\Psi_{n-1})L_n$$

where

$$L_n \;=\; \begin{pmatrix} q^{n+1}\,\dfrac{E}{1-q^{2(n+1)}} & , & 1 \\ \\ -q\,\dfrac{1-q^{2n}}{1-q^{2(n+1)}} & , & 0 \end{pmatrix} \tag{5.5.6}$$

So that

$$(\Psi_{n+1},\Psi_n) = (1,\;0)L_1\;L_2\;\ldots L_n \tag{5.5.7}$$

Since

$$L_\infty \;=\; \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}\;,\quad L_\infty^2 \;=\; \begin{pmatrix} -q & 0 \\ 0 & -q \end{pmatrix} \tag{5.5.8}$$

this product is convergent, and one can introduce its limits in the form

$$\begin{array}{ll} \Psi_{4n} \longrightarrow q^{2n}\;\Phi_0(E) & \quad \Psi_{4n+2} \longrightarrow -q^{2n+1}\;\Phi_0(E) \\ \Psi_{4n+1} \longrightarrow q^{2n}\;\Phi_1(E) & \quad \Psi_{4n+3} \longrightarrow -q^{2n+1}\;\Phi_1(E) \end{array} \tag{5.5.9}$$

Let us emphasise the dependence of a solution of (5.5.4) on E by notation

$$\Psi_n \;=\; \Psi_n(E) \tag{5.5.10}$$

Consider the truncated series

$$|\Psi_N\rangle \;=\; \sum_{n=0}^{N} |n\rangle \Psi_n(E)\;, \quad \text{and} \quad \langle\Psi'_N| \;=\; \sum_{n=0}^{N} \Psi_n(E')\langle n| \tag{5.5.11}$$

One obtains for these series the following self-consistence condition,

$$\Big(\langle\Psi'_N|\boldsymbol{\mathcal{H}}\Big)|\Psi_N\rangle \;=\; \langle\Psi'_N|\Big(\boldsymbol{\mathcal{H}}|\Psi_N\rangle\Big) \tag{5.5.12}$$

where we assume the action of $\boldsymbol{\mathcal{H}}$ to the left and to the right respectively. Then, equation (5.5.12) produces

$$\sum_{n=0}^{N-1} \Psi'_n\Psi_n = q^{-N}(1-q^{2N})\,\frac{\Psi'_N\Psi_{N-1} - \Psi'_{N-1}\Psi_N}{E'-E} \tag{5.5.13}$$

and therefore

$$\sum_{n=0}^{\infty} \Psi_n(E^{'})\Psi_n(E) = q^{-1}\,\frac{\Phi_1(E^{'})\Phi_0(E) - \Phi_0(E^{'})\Phi_1(E)}{E^{'}-E} \tag{5.5.14}$$

Thus, in the case of $\boldsymbol{\mathcal{U}_q(sl_2)}$ and Fock space representation, we expect a continuous spectrum of $\boldsymbol{\mathcal{H}}$ similarly to the classical case of $\boldsymbol{sl_2}$.

It worth to mention here another variant of the Hamiltonian. Namely, let now

$$\boldsymbol{E} \;=\; q^N\,\boldsymbol{e}\;, \quad \boldsymbol{F} \;=\; \boldsymbol{f}\,q^N \tag{5.5.15}$$

They satisfy q-deformed commuting relation

$$q\boldsymbol{E}\boldsymbol{F} \;-\; q^{-1}\boldsymbol{F}\boldsymbol{E} \;=\; (q-q^{-1})(\boldsymbol{k}^2-1) \tag{5.5.16}$$

Alternative Hamiltonian is

$$\boldsymbol{\mathcal{H}}' \;=\; \frac{\boldsymbol{E}+\boldsymbol{F}}{\mathsf{i}} \tag{5.5.17}$$

The recursion relation for the corresponding eigenvalue problem

$$(1-q^{2n+2})\Psi_{n+1} \;+\; (1-q^{2n})\Psi_{n-1} \;=\; E'\Psi_n \tag{5.5.18}$$

has the explicit solution,

$$\Psi_n \;=\; A\,\Phi_n(\omega) \;+\; B\,\Phi_n(\omega^{-1}) \tag{5.5.19}$$

where

$$E' \;=\; \omega \;+\; \omega^{-1} \tag{5.5.20}$$

and

$$\Phi_n(\omega) \;=\; \omega^n \sum_{k=0}^{\infty} \frac{(-\omega^2 q^2; q^4)_k}{(q^2; q^2)_k (\omega^2 q^2; q^2)_k}\, q^{2k(n+1)} \tag{5.5.21}$$

Analytical properties of this wave function, its singular initial condition $\Psi_{-1} = 0$, and the quantisation procedure requires further investigation.

5.6 Modular representation

Advantage of q-deformation is the extensives of the representation theory. Here we describe briefly another class of representations of $\boldsymbol{\mathcal{U}_q(sl_2)}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the generators of the simple Weyl algebra,

$$\boldsymbol{uv} = q^2 \boldsymbol{vu} \tag{5.6.1}$$

Generators of $\boldsymbol{\mathcal{U}_q(sl_2)}$ can be defined by

$$\begin{cases} \boldsymbol{e} = \boldsymbol{v}(z - q\boldsymbol{u}) \\ \boldsymbol{f} = \boldsymbol{v}^{-1}(z^{-1} - q\boldsymbol{u}^{-1}) \\ \boldsymbol{k} = \boldsymbol{u}\ , \end{cases} \tag{5.6.2}$$

where z $\in \mathbb{C}$ is a parameter. Relations

$$[\boldsymbol{e}, \boldsymbol{f}] = (q - q^{-1})(\boldsymbol{k} - \boldsymbol{k}^{-1}) \tag{5.6.3}$$

and

$$\boldsymbol{ke} = q^2 \boldsymbol{ek}, \qquad \boldsymbol{kf} = q^{-2} \boldsymbol{fk} \tag{5.6.4}$$

can be verified straightforwardly.

The q-Casimir operator (5.4.5) for (5.6.2) has the value

$$\boldsymbol{C} = z + z^{-1} \tag{5.6.5}$$

To define representation of (5.6.2), let us re-parametrise

$$q^2 \;=\; \boldsymbol{e}^{2\pi \mathrm{i} b^2}\ , \quad b^2 \;=\; \boldsymbol{e}^{\mathrm{i}\theta}\ , \quad |q^2| < 1 \tag{5.6.6}$$

and

$$\boldsymbol{u} \;=\; \boldsymbol{e}^{2\pi b\boldsymbol{x}} \;, \quad \boldsymbol{v} = \boldsymbol{e}^{2\pi b\boldsymbol{p}} \tag{5.6.7}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ is the canonical Heisenberg pair,

$$[\boldsymbol{x},\ \boldsymbol{p}] = \frac{\mathsf{i}}{2\pi} \tag{5.6.8}$$

Hermitian conjugation of $\boldsymbol{u}$ and $\boldsymbol{v}$ gives the modular counterpart of the modular double,

$$\bar{\boldsymbol{u}} = \boldsymbol{e}^{2\pi b^{-1}\boldsymbol{x}} \;, \quad \bar{\boldsymbol{v}} = \boldsymbol{e}^{2\pi b^{-1}\boldsymbol{p}} \;, \quad \bar{q}^{\;2} = \boldsymbol{e}^{-2\pi \mathsf{i} b^{-1}} \;, \quad \bar{\boldsymbol{u}}\bar{\boldsymbol{v}} = \bar{q}^{\;-2}\bar{\boldsymbol{v}}\bar{\boldsymbol{u}} \tag{5.6.9}$$

The conjugated Weyl elements form the modular counterpart of $\mathcal{U}_q(sl_2)$,

$$\begin{cases} \bar{\boldsymbol{e}} = (1 - \bar{q}\bar{z}\bar{\boldsymbol{u}})\bar{\boldsymbol{v}} \\ \bar{\boldsymbol{f}} = (\bar{z}^{-1} - \bar{q}\bar{\boldsymbol{u}}^{-1})\bar{\boldsymbol{v}}^{-1} \\ \bar{\boldsymbol{k}} = \bar{\boldsymbol{u}} \end{cases} \tag{5.6.10}$$

By construction, elements $\boldsymbol{u}, \boldsymbol{v}$ commute with their modular counterparts $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$. The union of (5.6.2) and (5.6.4) with (5.6.7) and (5.6.9) is called the modular double $\boldsymbol{\mathcal{U}_{q,\bar{q}}(sl_2)}$.

Representation of the modular double is equivalent to the representation of Heisenberg algebra with some extra analyticity conditions. Namely,

$$\boldsymbol{u}|x\rangle = |x\rangle \boldsymbol{e}^{2\pi b x} \;, \quad \boldsymbol{v}|x\rangle = |x + \mathsf{i}b\rangle \tag{5.6.11}$$

and

$$\langle x|\boldsymbol{u} = \boldsymbol{e}^{2\pi b x}\langle x| \;, \quad \langle x|\boldsymbol{v} = \langle x - \mathsf{i}b| \tag{5.6.12}$$

Analyticity conditions mentioned above are that a wave function $\Psi(x) \;=\; \langle x|\Psi\rangle$ must be entire in the whole complex plane $x \in \boldsymbol{\mathcal{C}}$.

The concept of self-adjoint Hamiltonians now is replaced by a concept of normal operators. In particular, one can define

$$\boldsymbol{\mathcal{H}} = \boldsymbol{e} + \boldsymbol{f} \quad \text{and} \quad \bar{\boldsymbol{\mathcal{H}}} = \bar{\boldsymbol{e}} + \bar{\boldsymbol{f}} \tag{5.6.13}$$

where $\boldsymbol{\mathcal{H}}$ and $\bar{\boldsymbol{\mathcal{H}}}$ are normal,

$$[\boldsymbol{\mathcal{H}},\ \bar{\boldsymbol{\mathcal{H}}}] = 0 \tag{5.6.14}$$

Problem of simultaneous diagonalization of $\boldsymbol{\mathcal{H}}$ and $\bar{\boldsymbol{\mathcal{H}}}$ takes the place of usual Stationary Schrödinger equation,

$$\boldsymbol{\mathcal{H}}|\Psi\rangle = |\Psi\rangle E \quad \text{and} \quad \bar{\boldsymbol{\mathcal{H}}}|\Psi\rangle = |\Psi\rangle \bar{E} \tag{5.6.15}$$

In coordinate representation,

$$\langle x|\boldsymbol{\mathcal{H}}|\Psi\rangle \;=\; \langle x|\Psi\rangle E \tag{5.6.16}$$

the first equation becomes

$$(1 - q^{-1}u)\Psi(-\mathrm{i}b) + (-q^{-1}u^{-1})\Psi(x + \mathrm{i}b) = E\Psi(x) \tag{5.6.17}$$

where

$$u = e^{2\pi bx} \tag{5.6.18}$$

Its modular counterpart is

$$(1 - \bar{q}\bar{z}\bar{u})\Psi(x - ib^{-1}) + (\bar{z}^{-1} - \bar{q}\bar{u}^{-1})\Psi(x + ib^{-1}) = \bar{E}\Psi(x) \tag{5.6.19}$$

This pair of modular equations was considered in [5]. However, this case requires further investigation.

Chapter 6

Discussion

6.1 Introduction

We have presented a discussion about the theory of classical Lie algebra and representation theory of quantum algebra $\mathcal{U}_q(sl_2)$ which were considered in a certain infinity dimensional representation corresponding to the lowest weight 1. This chapter represents an elaboration of a summary of the discussion, followed by the presentation of major findings we have found out and a review of similar studies.

6.2 Summary

We have taken in consideration the representation theory of classical Lie algebra and representation theory of quantum algebra $\mathcal{U}_q(sl_2)$. We have analysed an equivalent module of the Fock space representation of the quantum oscillator. We have proved that this quantum operator has continuous spectrum.

Chapter1 has presented the basic principles to understand the general idea of this paper. We have looked at the theory of the classical Lie algebra and quantum group $\mathcal{U}_q(sl_2)$ including the fundamental principles of quantum mechanics. We have considered the observables and Born rule principles of measurement theory and the fundamental relationship between quantum mechanics and representation theory. We have debated of the theory of unitary group representations and the symmetry groups representations on function spaces. We have also scrutinized the quantum groups solvable models. The goal was to present a sufficient algebraic basis for entering this exciting world which is pregnant with possibility and has a richness of theory promising to lead to ever greater discoveries.

Chapter 2 has been concerned by the Lies algebra and their representation theory. We have looked at the group SO(3) and its algebra. We have also reviewed the angular momentum in quantum mechanics, including Pauli matrices and nicknames, Casimir operator, the spirit of quantum mechanics, and the fields in quantum theories. We have disputed over examples and Fock space of $\boldsymbol{sl_2}$ representation theory. We have also dialogued on simple lie algebras and their common features, the algebras of $\boldsymbol{sl_n}$, and the Carton-Weyl basis theorem.

Chapter 3 included representation theory in classical algebra. We looked at the lie algebra $\boldsymbol{sl_2}$ in specific infinitely dimensional representation with lowest weight 1. The module of representation is equivalent to the Fock space representation of the quantum oscillator. We took into consideration that the stationary Schrödinger equation for the self conjugated Hamiltonian $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathrm{i}}$ where $\boldsymbol{e}$ and $\boldsymbol{f}$ are creating and annihilating operators for the algebra $\boldsymbol{sl_2}$ considered for infinite dimensional representation with the lowest weight equal to 1, which is equivalent to the Fock space. We paid attention to the analysis of our recursion. The general expression of the $\Psi_n(E)$ include the following functions: $A(E), \Psi(E), \delta_n(E), E_n(E)$ as you see in (3.3.13). We gave the strict method for defining $\delta(n, E)$ and $E(n, E)$ is analytically in the forms of series expansion with respect to γ_n and E, however the function $A(E)$ and $\Psi(E)$ are numerically defined only for real E.

Chapter 4 is about $\boldsymbol{\mathcal{U}_q(sl_2)}$ and its representations. We first looked at finite dimensional representations of $\boldsymbol{\mathcal{U}}$ and then determines the center of $\boldsymbol{\mathcal{U}}$. In both cases the result depends very much on whether q is a root of unity or not. The general pattern to evolve is this: If q is not a root of unity, then $\boldsymbol{\mathcal{U}}$ behaves like the enveloping algebra of $\boldsymbol{sl_2}$ over a field of characteristic 0; If q is a root of unity, then $\boldsymbol{\mathcal{U}}$ behaves like the enveloping algebra of $\boldsymbol{sl_2}$ over a field of prime characteristic. And this statement holds independently of the characteristic with a small exception if $\boldsymbol{k}$ has characteristic 2. But the really exciting feature of this situation is this: If we take $\boldsymbol{k} = \mathbb{C}$ and q equal to a primitive p_th root of unity with p a prime, $p \geq 3$, then we got a representation theory over $\mathbb{C}$ that looks like the representation theory of $\boldsymbol{sl_2}$ over an algebraically closed field of characteristic p.

Chapter 5 involved representation theory of $\boldsymbol{sl_2}, \boldsymbol{\mathcal{U}_q(sl_2)}$, oscillator, and q-oscillator. We supposed algebra $\boldsymbol{sl_2}$ by three elements $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ satisfying the known relations (5.3.1). The quantum oscillator algebra $\mathcal{O}$ is generated by operator $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$. The quantum oscillator algebra admits the representation over the infinite-dimensional right module called the Fock space, where it is discussed in first section in this chapter. In representation theory of $\boldsymbol{sl_2}$, we focused on right and left module for (5.3.1) using quantum mechanical approach. So, we mentioned to the lowest weight of the corresponding right and left module, the finite dimensional representation and Casimir operator. Then, we moved to $\boldsymbol{\mathcal{U}_q(sl_2)}$ and q-oscillator, where we reviewed the free module $\mathbb{M}_{\lambda_0}$, Casimir operator, representation over Fock space, normalization changes of the states for finite dimensional representation of $\boldsymbol{\mathcal{U}_q(sl_2)}$ and the right co_module.

We turned to Hamiltonians in $\boldsymbol{\mathcal{U}_q(sl_2)}$. Our stationary Schrödinger equation for the self conjugates Hamiltonian is considered $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$ with its eigenvalue probed in $\boldsymbol{\mathcal{H}}|\Psi\rangle = |\Psi\rangle E$. We gave detailed analysis for a solution of the recursion (5.4.4) . The last point in this chapter is modular representation where we demonstrated briefly another division of representations of $\boldsymbol{\mathcal{U}_q(sl_2)}$. We supposed $\boldsymbol{u}$ and $\boldsymbol{v}$ be the generators of the simple Weyl algebra, $\boldsymbol{uv} = \boldsymbol{q}^2\boldsymbol{vu}$.

6.3 Major findings

6.3.1 Major finding-1

Algebra $\boldsymbol{sl_2}$ and its representations:
We considered the algebra $\boldsymbol{sl_2}$ generated by the following operators: $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$

$$[\boldsymbol{e},\boldsymbol{f}] = \boldsymbol{h} \quad , \quad [\boldsymbol{h},\boldsymbol{e}] = 2\boldsymbol{e} \quad , \quad [\boldsymbol{h},\boldsymbol{f}] = -2\boldsymbol{f}$$

In an infinite dimension representation

$$\boldsymbol{e}|n\rangle = |n+1\,\rangle\, \mathsf{i}(n+1) \quad , \quad \boldsymbol{f}|n\rangle = |n-1\,\rangle\, \mathsf{i}n \quad , \quad \boldsymbol{h}|n\rangle = |n\,\rangle(2n+1)$$

This representation is the representation with the lowest weight + 1 (in physics, it is called "spin_ γ_2"). The module is called the Fock space, its co_module is defined by

$$\langle n|n^{'}\rangle\delta_{n,n^{'}} \quad , \quad n\; n^{'} \geq 0$$

and, feature of representation is that it is not unitary: $\boldsymbol{e}^{\dagger} = -\boldsymbol{f}$

Stationary Schrödinger equation:

Our consideration is self-conjugated unbounded Hamiltonian $\boldsymbol{\mathcal{H}} = \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$ and the stationary Schrödinger equation for it is $\boldsymbol{\mathcal{H}}|\Psi\rangle = |\Psi\rangle E$.

Then, we got our recursion like this:

$$(n+1)\psi_{n+1} + n\psi_{n-1} = E\psi_n$$

where n ≥ 0 and where we temporarily assume $\psi_0 = 1$.

The recursion:
E for now is free parameter. So, we assumed implicitly $|\psi\rangle = |\psi_E\rangle$ and $\psi_n(E)$. We wrote our recursion in matrix form $(\psi_n, \psi_{n+1}) = \psi_{n-1}, \psi_n \cdot L_{n+1}$ where

$$L_n = \begin{pmatrix} 0 & -1+\frac{1}{n} \\ 1 & \frac{E}{n} \end{pmatrix}$$

Thus, the format solution to the Eigenvector problem is $(\psi_{n-1}, \psi_n) = (0,1)L_1 \cdot L_2 \cdots L_{n-1} \cdot L_n$ where $L_n = L_n(E)$.

After that we moved to analysis of our recursion diagonalising matrix L_n,

$$L_n = P_n^{-1} \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^* \end{pmatrix}$$

Then, we could deduce the following asymptotic

$$\psi_n = \frac{A_n}{\sqrt{n}} \cos\left(\frac{E}{2}\log n - \frac{\pi n}{2} + \varphi_n\right), \quad n \geq 1$$

Here,

$$A_n = A(E)\left(1 + \delta_n(E)\right), \quad \varphi_n = \varphi(E) + \epsilon_n(E),$$

precise form of the symptotic corrections is the following,

$$\delta_n(E) \;=\; \frac{1}{4\,n} + \frac{2E^2+1}{32\,n^2} - \frac{5(2E^2-1)}{128\,n^3} + \mathcal{O}(n^{-4})$$

and

$$\epsilon_n(E) \;=\; \frac{E}{4\,n} - \frac{E(E^2-5)}{96\,n^2} + \frac{E(E^2-9)}{96\,n^3} + \mathcal{O}(n^{-4})$$

We deduce that $\delta_n(E)$ and $\epsilon_n(E)$ can be produced from our recursion by a boot step up to any power of $\frac{1}{n}$.

Orthogonality:
We considered a truncated state, $|\psi_E^{(N)}\rangle \;=\; \sum_{n=0}^{N} |n\rangle \psi_n(E)$, where $\psi^{(N)}$ is defined by our recursion, then, straightforward computation becomes

$$\boldsymbol{\mathcal{H}}\,|\psi_E^{(N)}\rangle \;=\; |\psi_E^{(N-1)}\rangle\; E \;+\; |N\rangle\; N\psi_{N-1}(E) \;+\; |N+1\rangle\; (N+1)\psi_N(E)$$

We considered

$$\langle\psi_{E'}^{(N)}|\;\boldsymbol{\mathcal{H}}\;|\psi_E^{(N)}\rangle$$

Then we got

$$\langle\psi_{E'}^{(N-1)}|\psi_E^{(N-1)}\rangle \;=\; \frac{N}{E-E'}\;\left(\psi_N(E)\psi_{N-1}(E') - \psi_N(E')\psi_{N-1}(E)\right)$$

Assuming our asymptotic for ψ_N, we found that,

$$\langle\psi_{E'}^{(N)}|\psi_E^{(N)}\rangle \;=\; A(E')A(E)\frac{\sin\left(\dfrac{E'-E}{2}\log N + \varphi(E') - \varphi(E)\right)}{E'-E}$$

Therefore $N \to \infty$, then we obtained $\langle\psi_{E'}|\psi_E\rangle \;=\; \pi A(E)^2\delta(E-E')$.
Also, here we found that E' $\to$ E is singular.

6.3.2 Major finding-2

We considered the algebra $\boldsymbol{sl_2}$ generated by operators $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ satisfying the relation $[\boldsymbol{e},\boldsymbol{f}] = \boldsymbol{h}$, $[\boldsymbol{h},\boldsymbol{e}] = 2\boldsymbol{e}$, $[\boldsymbol{h},\boldsymbol{f}] = -2\boldsymbol{f}$. The quantum oscillator algebra $\mathcal{O}$ is generated by operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$.

The state $|0\rangle$ is the Fock vacuum, annihilated by $\boldsymbol{a}$, the other states are the results of application of creation operators to the vacuum

$$\boldsymbol{a}|0\rangle = 0 \ , \quad |n\rangle = \frac{1}{\sqrt{n!}}(\boldsymbol{a}^\dagger)^n|0\rangle$$

In our case, we obtained explicit homomorphism from oscillator algebra to $\boldsymbol{sl_2}$ like this:

$$\begin{cases} \boldsymbol{e} = \mathsf{i}\sqrt{N}\boldsymbol{a}^\dagger = \mathsf{i}\boldsymbol{a}^\dagger\sqrt{N+1} \\ \boldsymbol{f} = \mathsf{i}\sqrt{N+1}\boldsymbol{a} = \mathsf{i}\boldsymbol{a}\sqrt{N} \\ \boldsymbol{h} = 2N+1 \end{cases}$$

After that, we went to representation theory of $\boldsymbol{sl_2}$. We defined the whole right module by

$$|\psi_n\rangle \ = \ \boldsymbol{e}^n|\psi_0\rangle$$

and its co-module by

$$\langle\bar{\psi}_m|\psi_n\rangle \ = \ \delta_{m,n}$$

The Casimir operator is defined by

$$\boldsymbol{C} \ = \ \boldsymbol{e}\boldsymbol{f} + \frac{1}{4}\,\boldsymbol{h}^2 - \frac{1}{2}\,\boldsymbol{h}$$

and its Eigenvalue on $\mathbb{M}_{\lambda_0}$ is

$$\boldsymbol{C}\ |\psi_n\rangle \ = \ |\psi_n\rangle\,\frac{\lambda_0(\lambda_0-2)}{4} \ , \quad \forall\, n$$

We obtained that $N+1$ dimensional representation M_N and

$$\mathbb{M}_N \ = \ \mathbb{V}_N \ \oplus \ \mathbb{M}_{-N-1}$$

We did some change of normalization:

$$|\phi_n\rangle = |\psi_n\rangle\ C_n \ , \quad \langle\bar{\phi}_n| = \frac{1}{C_n}\ \langle\bar{\psi}|$$

Then,

$$C_n = C_0 \sqrt{\frac{(N-n)!}{n!N!}} \quad and \quad \boldsymbol{f} = \boldsymbol{e}^\dagger$$

We considered our case:

$$\lambda_0 = 1 \qquad (\text{spin} = -\frac{1}{2})$$

Then, with another normalization

$$|\phi_n\rangle \;=\; |\psi_n\rangle \, C_n \;, \qquad \langle\bar{\phi}_n| \;=\; \frac{1}{C_n} \, \langle\bar{\psi}_n|$$

So, that

$$C_n \;=\; \frac{1}{\mathsf{i}^n n!}$$

In the new basis one has

$$\boldsymbol{e}\,|\phi_n\rangle \;=\; |\phi_{n+1}\rangle \, \mathsf{i}\,(n+1) \quad | \quad \langle\bar{\phi}_n|\,\boldsymbol{f} \;=\; \mathsf{i}\,(n+1)\,\langle\bar{\phi}_{n+1}|$$

$$\boldsymbol{f}\,|\phi_n\rangle \;=\; |\phi_{n-1}\rangle \, \mathsf{i}\,n \quad | \quad \langle\bar{\phi}_n|\,\boldsymbol{e} \;=\; \mathsf{i}\,n\,\langle\bar{\phi}_{n-1}|$$

Then, we worked with $\boldsymbol{\mathcal{U}_q(sl_2)}$ where it is defined by $\boldsymbol{e}$, $\boldsymbol{f}$, and $\boldsymbol{k}$,

$$[\boldsymbol{e},\boldsymbol{f}] \;=\; (q-q^{-1})\,(\boldsymbol{k}-\boldsymbol{k}^{-1}) \;; \;\; \boldsymbol{k}\boldsymbol{e} = q^2\boldsymbol{e}\boldsymbol{k} \;; \;\; \boldsymbol{k}\boldsymbol{f} = q^{-2}\boldsymbol{f}\boldsymbol{k}$$

We reviewed some representations theory for $\boldsymbol{\mathcal{U}_q(sl_2)}$, then we looked at our case

$$\lambda_0 = 1 \qquad (\text{spin} = -\frac{1}{2})$$

The normalization is

$$|\phi_n\rangle \;=\; |\psi_n\rangle \, C_n$$

So that,

$$C_n \;=\; \frac{q^{n(n+1)/2}}{\mathsf{i}^n (q^2;q^2)_n}$$

and,

$$\begin{cases} \boldsymbol{e}|\phi_n\,\rangle = |\phi_{n+1}\,\rangle \;\mathsf{i}q^{-n-1}(1-q^{2(n+1)}) \\ \boldsymbol{f}|\phi_n\,\rangle = |\phi_{n-1}\,\rangle \;\mathsf{i}q^{-n}(1-q^{2n}) \\ \boldsymbol{k}|\phi_n\,\rangle = |\phi_n\,\rangle \; q^{2n+1} \end{cases}$$

The right co-module, defined by $\langle\bar{\phi}_n|\phi_n\rangle = \delta_{m,n}$, has the symmetric action for $\boldsymbol{e}, \boldsymbol{f}$:

$$\langle\bar{\phi}_n|\boldsymbol{e} = \mathrm{i}q^{-n}(1-q^{2n})\ \langle\bar{\phi}_{n-1}| \quad , \quad \langle\bar{\phi}_n|\boldsymbol{f} = \mathrm{i}q^{-n-1}(1-q^{2(n+1)})\ \langle\bar{\phi}_{n+1}|$$

So,

$$\boldsymbol{e}^T \ = \ \boldsymbol{f} \qquad \text{but} \qquad \boldsymbol{e}^\dagger \ = \ -\boldsymbol{f}$$

The homomorphism is

$$\boldsymbol{e} \ = \ \mathrm{i}\,\boldsymbol{a}^+\ (q\boldsymbol{k})^{-1/2}\sqrt{1-q\boldsymbol{k}}\ , \qquad \boldsymbol{f} \ = \ \mathrm{i}\ (q\boldsymbol{k})^{-1/2}\sqrt{1-q\boldsymbol{k}}\ \boldsymbol{a}^-$$

The element $\boldsymbol{k}$ is the same in both cases,
Then,

$$|\phi_n\rangle \ \equiv \ |n\rangle\ , \qquad \langle\bar{\phi}_n| \ \equiv \ \langle n|$$

We turned in Hamiltonian with $\boldsymbol{\mathcal{U}_q(sl_2)}$.

Our self-conjugated Hamiltonian is considered by

$$\boldsymbol{H} \ = \ \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathrm{i}}$$

and the stationary Schrödinger equation for it is

$$\boldsymbol{\mathcal{H}}\ |\Psi\rangle \ = \ |\Psi\rangle\ E$$

So, we obtained the following recursion:

$$q^{-n}(1-q^{2n})\Psi_{n-1} + q^{-n-1}(1-q^{2(n+1)})\ \Psi_{n+1} = E\Psi_n$$

Analysis of the recursion:

$$(\Psi_{n+1}, \Psi_n) = (\Psi_n, \Psi_{n-1})L_n$$

So that,

$$(\Psi_{n+1}, \Psi_n) = (1,\ 0)L_1\ L_2\ \ldots L_n$$

We supposed the truncated series.

$$|\Psi_N\rangle \ = \ \sum_{n=0}^{N}|n\rangle\Psi_n(E) \quad \text{and} \quad \langle\Psi'_N| \ = \ \sum_{n=0}^{N}\Psi_n(E')\langle n|$$

One got for these series the following self-consistence condition,

$$\langle \Psi'_N | \Big(\boldsymbol{\mathcal{H}} | \Psi_N \rangle \Big) \; = \; \Big(\langle \Psi'_N | \boldsymbol{\mathcal{H}} \Big) | \Psi_N \rangle$$

So, That

$$\sum_{n=0}^{\infty} \Psi_n(E^{'}) \Psi_n(E) = q^{-1} \, \frac{\Phi_1(E^{'})\Phi_0(E) - \Phi_0(E^{'})\Phi_1(E)}{E' - E}$$

Thus, in the case of $\boldsymbol{\mathcal{U}_q(sl_2)}$ and Fock space representation could be a continuous spectrum of $\boldsymbol{\mathcal{H}}$ similarly to situation of $\boldsymbol{sl_2}$.

We indicated another variant of the Hamiltonian.

$$\boldsymbol{E} \; = \; q^N \, \boldsymbol{e} \; , \quad \boldsymbol{F} \; = \; \boldsymbol{f} \, q^N \; , \quad q\boldsymbol{E}\boldsymbol{F} \; - \; q^{-1}\boldsymbol{F}\boldsymbol{E} \; = \; (q - q^{-1})(\boldsymbol{k}^2 - 1)$$

and,

$$\boldsymbol{\mathcal{H}}' \; = \; \frac{\boldsymbol{E} + \boldsymbol{F}}{\mathsf{i}}$$

For that, our recursion,

$$(1 - q^{2n+2})\Psi_{n+1} \; + \; (1 - q^{2n})\Psi_{n-1} \; = \; E'\Psi_n$$

has the following solution,

$$\Psi_n \;\; = \;\; A \, \Phi_n(\omega) \; + \; B \, \Phi_n(\omega^{-1})$$

Here we referred to another kind of representations of $\boldsymbol{\mathcal{U}_q(sl_2)}$. We let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the generators of the simple Weyl algebra,

$$\boldsymbol{u}\boldsymbol{v} = q^2 \boldsymbol{v}\boldsymbol{u}$$

We defined generators of $\boldsymbol{\mathcal{U}_q(sl_2)}$ by

$$\boldsymbol{e} = \boldsymbol{v}(1 - q\boldsymbol{u}) \quad , \quad \boldsymbol{f} = \boldsymbol{v}^{-1}(z^{-1} - q\boldsymbol{u}^{-1}) \quad , \quad \boldsymbol{k} = \boldsymbol{u}$$

Relations:

$$[\boldsymbol{e}, \boldsymbol{f}] = (q - q^{-1})(\boldsymbol{k} - \boldsymbol{k}^{-1}) \; , \quad k\boldsymbol{e} = q^2 \boldsymbol{e}\boldsymbol{k}, \quad , \quad \boldsymbol{k}\boldsymbol{f} = q^{-2}\boldsymbol{f}\boldsymbol{k}$$

The q-Casimir operator has the value

$$\boldsymbol{C} = z + z^{-1}$$

To define representation of generators of $\boldsymbol{\mathcal{U}_q(sl_2)}$, then we had,

$$q^2 \;=\; \boldsymbol{e}^{2\pi \mathsf{i} b^2} \;, \quad b^2 \;=\; \boldsymbol{e}^{\mathsf{i}\theta} \;, \quad |q^2| < 1 \;, \quad \boldsymbol{u} \;=\; \boldsymbol{e}^{2\pi b\boldsymbol{x}} \;, \quad \boldsymbol{v} = \boldsymbol{e}^{2\pi b\boldsymbol{p}}$$

here $\boldsymbol{x}$ and $\boldsymbol{p}$ is the canonical Heisenberg pair, so that

$$[\boldsymbol{x},\ \boldsymbol{p}] = \frac{\mathsf{i}}{2\pi}$$

Hermitian conjugation of $\boldsymbol{u}$ and $\boldsymbol{v}$ gives the modular counterpart of the modular double,

$$\bar{\boldsymbol{u}} = \boldsymbol{e}^{2\pi b^{-1}\boldsymbol{x}} \;, \quad \bar{\boldsymbol{v}} = \boldsymbol{e}^{2\pi b^{-1}\boldsymbol{p}} \;, \quad \bar{q}^{\;2} = \boldsymbol{e}^{-2\pi \mathsf{i} b^{-1}} \;, \quad \bar{\boldsymbol{u}}\bar{\boldsymbol{v}} = \bar{q}^{\;-2}\bar{\boldsymbol{v}}\bar{\boldsymbol{u}}$$

The conjugated Weyl elements form the modular counterpart of $\mathcal{U}_q(sl_2)$,

$$\bar{\boldsymbol{e}} = (1 - \bar{q}\bar{z}\bar{\boldsymbol{u}})\bar{\boldsymbol{v}} \;, \quad \bar{\boldsymbol{f}} = (\bar{z}^{-1} - \bar{q}\bar{\boldsymbol{u}}^{-1})\bar{\boldsymbol{v}}^{-1}, \quad \bar{\boldsymbol{k}} = \bar{\boldsymbol{u}}$$

We defined the normal operators by

$$\boldsymbol{\mathcal{H}} = \boldsymbol{e} + \boldsymbol{f} \quad \text{and} \quad \bar{\boldsymbol{\mathcal{H}}} = \bar{e} + \bar{\boldsymbol{f}}$$

The problem of simultaneous diagonalization of $\boldsymbol{\mathcal{H}}$ and $\bar{\boldsymbol{\mathcal{H}}}$ has

$$\boldsymbol{\mathcal{H}}|\Psi\rangle = |\Psi\rangle E \quad \text{and} \quad \bar{\boldsymbol{\mathcal{H}}}|\Psi\rangle = |\Psi\rangle \bar{E} \quad \text{and} \quad \langle x|\boldsymbol{\mathcal{H}}|\Psi\rangle \;=\; \langle x|\Psi\rangle E$$

Then, we got the following equation

$$(1 - q^{-1}u)\Psi(-\mathsf{i}b) + (-q^{-1}u^{-1})\Psi(x + \mathsf{i}b) = E\Psi(x)$$

Its modular counterpart was

$$(1 - \bar{q}\bar{z}\bar{u})\Psi(x - ib^{-1}) + (\bar{z}^{-1} - \bar{q}\bar{u}^{-1})\Psi(x + ib^{-1}) = E\Psi(x)$$

6.4 Discussion of similar studies

The similar studies are based on Sergeev's (2005) observation [4] that the asymptotic of the wave function for difference modular Schrödinger-type equations is not related to the spectral problem. Instead, he formulated

Proposition 6.1.
The quantisation condition for modular difference equations is the entireness *of the wave function $\psi(x)$ in the whole complex plane $x \in \mathbb{C}$.*

6.4.1 Study-1

Kashaev and Sergey (2017) developed this idea in their paper "On Spectral equations for the Modular Oscillator" [5]. They discussed the spectral of the following commuting operators:

$$\boldsymbol{H} = \boldsymbol{v} + \boldsymbol{v}^{-1} + \boldsymbol{u} + \boldsymbol{u}^{-1} \quad , \quad \bar{\boldsymbol{H}} = \bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}^{-1} + \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}^{-1}$$

These Hamiltonians establish the direct relation of Kashaev and Sergeev work with our study since

$$\boldsymbol{H} \;=\; \boldsymbol{e} + \boldsymbol{f} \;, \quad \boldsymbol{e} \;=\; \boldsymbol{v} + \boldsymbol{u} \;, \quad \boldsymbol{f} \;=\; \boldsymbol{v}^{-1} + \boldsymbol{u}^{-1} \tag{6.4.1}$$

where $\boldsymbol{e}, \boldsymbol{f}$ are the $\mathcal{U}_q(sl_2)$ elements.

The Schrödinger equations

$$\bar{\boldsymbol{H}} = \bar{\boldsymbol{H}}^{\dagger} \quad \text{and} \quad \boldsymbol{H}|\psi\rangle = \varepsilon|\psi\rangle \quad , \quad \bar{\boldsymbol{H}}|\psi\rangle = \bar{\varepsilon}|\psi\rangle$$

correspond to the following functional difference equations:

$$\begin{aligned} \psi(x + \mathsf{ib}) + \psi(x - \mathsf{ib}) \;&=\; (\varepsilon - 2\cosh(2\pi \mathsf{b} x))\psi(x) \\ \psi(x - \mathsf{ib}^{-1}) + \psi(x + \mathsf{ib}^{-1}) \;&=\; (\bar{\varepsilon} - 2\cosh(2\pi \mathsf{b}^{-1} x))\psi(x) \end{aligned} \tag{6.4.2}$$

where

$$u \;=\; \mathrm{e}^{2\pi \mathsf{b} x} \;, \quad q \;=\; \mathrm{e}^{\mathsf{i}\pi \mathsf{b}^2} \;, \quad \psi(x) \;=\; \langle x|\psi\rangle \tag{6.4.3}$$

and in the "strongly coupled regime" "bar" stands for the complex conjugation.

General solution $\psi(x) \in \mathbb{L}_2$ of (6.4.2) is given by

$$\psi(x) = \mathsf{b}^{-1}\ \mathsf{e}^{\pi \mathsf{i}\sigma 2 - \xi\pi\mathsf{i}/4}\ \mathsf{e}^{2\pi\eta x + \mathsf{i}\pi x^2}\ \ \frac{\overset{v}{\boldsymbol{\chi}}(u)\overline{\boldsymbol{\chi}(u)} + \xi\boldsymbol{\chi}(u)\overline{\overset{v}{\boldsymbol{\chi}}(u)}}{\theta_1(su,q)\theta_1(s^{-1}u,q)} \tag{6.4.4}$$

where

$$\eta \ = \ \frac{1}{2}(\mathsf{b}+\mathsf{b}^{-1})\ , \quad s \ = \ \mathrm{e}^{2\pi\mathsf{b}\sigma}\ , \quad \xi \in \{\pm 1\} \tag{6.4.5}$$

θ_1 is the Jacobi θ-function, and

$$\boldsymbol{\chi}(u) \ = \ \boldsymbol{\chi}_q(u;\varepsilon) \tag{6.4.6}$$

is the holomorphic solution of

$$f\big(\frac{u}{q^2}\big) + q^2u^2 f(q^2u) = (1-\varepsilon u + u^2) f(u) \tag{6.4.7}$$

It can be defined as follows. Let

$$L(u) \ = \ \begin{pmatrix} 1-\varepsilon u + u^2 & -q^2u^2 \\ 1 & 0 \end{pmatrix}\ , \quad M(u) \ = \ L(u)L(q^2u)L(q^4u)\cdots \tag{6.4.8}$$

Then,

$$M(u) \ = \ \begin{pmatrix} \boldsymbol{\chi}_q(q^{-2}u;\varepsilon) & 0 \\ \boldsymbol{\chi}_q(u;\varepsilon) & 0 \end{pmatrix} \tag{6.4.9}$$

The second solution to (6.4.7) is

$$\overset{v}{\boldsymbol{\chi}}_q(u;\varepsilon) \ = \ u^{-1}\boldsymbol{\chi}_q(u^{-1};\epsilon) \tag{6.4.10}$$

The wave function $\psi(x)$ has the properties

$$\psi(-x) = \xi\psi(x) \quad \textit{and} \quad \overline{\psi(x)} = \psi(x)$$

Parameter $s \ = \ \mathrm{e}^{2\pi\mathsf{b}\sigma}$ is related to ε by

$$\boldsymbol{\chi}(q^{-2}s;\varepsilon)\overset{v}{\boldsymbol{\chi}}_q(s;\varepsilon) - \boldsymbol{\chi}(s;\varepsilon)\overset{v}{\boldsymbol{\chi}}_q(q^{-2}s;\varepsilon) \ = \ 0 \tag{6.4.11}$$

so that

$$\varepsilon \ = \ \varepsilon(\sigma) \tag{6.4.12}$$

is implicitly defined function with rather complicated Riemann surface.

Quantisation condition for the spectral problem is

$$G(s,\varepsilon) \ = \ -\xi\,\overline{G(s,\varepsilon)} \tag{6.4.13}$$

where

$$G(u,\varepsilon) \;=\; \frac{\boldsymbol{\chi}_q(u;\varepsilon)}{\overset{v}{\boldsymbol{\chi}}_q(u;\varepsilon)} \tag{6.4.14}$$

Equation (6.4.13) has an infinite set of solutions

$$\varepsilon_k \;=\; \varepsilon(\sigma_k) \tag{6.4.15}$$

where σ_k lie on different sheets of the Riemann surface for $\varepsilon \;=\; \varepsilon(\sigma)$.

6.4.2 Study-2

The similar result was obtained by Kashaev R.M. and Sergeev S.M. (2019) in the paper called "On the spectrum of the Local P^2 Mirror Curve" [11]. They considered $\boldsymbol{u}$ and $\boldsymbol{v}$ be the Heisenberg–Weyl operators, the Hamiltonian,

$$\boldsymbol{H} \;=\; \boldsymbol{u} + \boldsymbol{v} + q^{-1}\boldsymbol{u}^{-1}\boldsymbol{v}^{-1} = \boldsymbol{u} + \boldsymbol{v} + q\boldsymbol{v}^{-1}\boldsymbol{u}^{-1} \tag{6.4.16}$$

and the Schrödinger equations

$$\boldsymbol{H}|\psi\rangle \;=\; \varepsilon|\psi\rangle\;, \quad \boldsymbol{H}^{\dagger}|\psi\rangle \;=\; \bar{\varepsilon}|\psi\rangle \tag{6.4.17}$$

The analysis of this Schrödinger equations is more complicated than the previous one. For instance, an analogue of $\boldsymbol{\chi}$ here is not *holomorphic*. However, all auxiliary functions can be presented as fastly convergent series of matrix products, the condition of *entireness* of the wave function is well posed, and the spectra of the Hamiltonians can be produced numerically.

Chapter 7

Conclusion and future work

7.1 Conclusion

This thesis represents a significant discussion on spectrum of some $\boldsymbol{sl_2}$ and $\boldsymbol{\mathcal{U}_q(sl_2)}$ related Hamiltonian in quantum mechanics. We opened the analysis by looking at basic principles for embracing the general idea of this thesis. We gave details on Lies algebra and their representation theory, thought over representation theory in classical algebra, scanned $\boldsymbol{\mathcal{U}_q(sl_2)}$) and its representations, and flirted with the representation theory of $\boldsymbol{sl_2}$, $\boldsymbol{\mathcal{U}_q(sl_2)}$, oscillator, and q-oscillator.

Specifically, we investigated the lie algebra $\boldsymbol{sl_2}$ in specific infinitely dimensional representation with lowest weight 1. We kept in mind that the stationary Schrödinger equation for the self conjugated Hamiltonian $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}$ where $\boldsymbol{e}$ and $\boldsymbol{f}$ were creating and annihilating operators for the algebra $\boldsymbol{sl_2}$ which was equivalent to the Fock space. We monitored keenly the analysis of our recursion (3.3.2). We delivered the strict method for defining $\delta(n, E)$ and $E(n, E)$ elected analytically in the forms of series expansion with respect to γ_n and E.

We probed representation theory of $\boldsymbol{sl_2}$, $\boldsymbol{\mathcal{U}_q(sl_2)}$, oscillator, and q-oscillator. We assumed algebra $\boldsymbol{sl_2}$ by three elements $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{h}$ propitiating the known relations (5.3.1). We remarked the lowest weight of the corresponding right and left module, the finite dimensional representation and Casimir operator. We maneuvered to $\boldsymbol{\mathcal{U}_q(sl_2)}$) and q-oscillator, where we went over the free module $\mathbb{M}_{\lambda_0}$, Casimir operator, representation over Fock space, normalization changes of the states for finite dimensional representation of $\boldsymbol{\mathcal{U}_q(sl_2)}$ and the right co-module. We shifted to

Hamiltonians in $\boldsymbol{\mathcal{U}_q(sl_2)}$. We handed over careful review for a solution of the recursion (3.3.2). In case of Fock space representation and $\boldsymbol{\mathcal{U}_q(sl_2)}$, it is a continuous spectrum of $\boldsymbol{H}$, similarly to the classical situation of $\boldsymbol{sl_2}$. Analytical features of our wave function, its singular initial condition $\Psi_{-1} = 0$. We exposed briefly another division of representations of $\boldsymbol{\mathcal{U}_q(sl_2)}$ where we deemed $\boldsymbol{u}$ and $\boldsymbol{v}$ be the generators of the simple Weyl algebra, $\boldsymbol{uv} = \boldsymbol{q}^2\boldsymbol{vu}$.

The ambition of this thesis was to expound a satisfactory algebraic basis for committing oneself into this exciting world which is loaded with possibility and bounty of theory auguring to instigate ever greater breakthroughs.

7.2 Further work

A first research project is to finalise the study of the Hamiltonians listed in the Section 5.4. Related project is the study of $\boldsymbol{sl_2}$ and $\boldsymbol{\mathcal{U}_q(sl_2)}$ Hamiltonians for reducible representations related to the co-product for both classical and q-deformed algebras.

A challenging problem is to construct the regimes and spectra of the most general Hamiltonians corresponding to a "genus 1 quantum curve" in the framework of modular doubles,

$$\boldsymbol{H} = \sum_{\alpha,\beta=-1,0,1} c_{\alpha,\beta}\, \boldsymbol{u}^{\alpha}\boldsymbol{v}^{\beta} \qquad c_{0,0} = 0 \tag{7.2.1}$$

Note that since $\boldsymbol{H}$ is unbounded, its normality regimes requires more detailed investigations.

A final problem in this direction is the construction of Bethe Ansatz for a modular $\mathcal{U}_{q,\overline{q}}(sl_2)$ spin chain. The Schrödinger equations considered here are the particular cases of more general $T - Q$ equations,

$$\begin{aligned} T(u)\,Q(x) &= a(u)Q(x+\mathsf{ib}) + b(u)Q(x-\mathsf{ib}) \\ \overline{T(u)}Q(x) &= \overline{a(u)}Q(x-\mathsf{ib}^{-1}) + \overline{b(u)}Q(x+\mathsf{ib}^{-1}) \end{aligned} \tag{7.2.2}$$

where T is an eigenvalue of the transfer matrix and Q is an eigenvalue of Baxter's Q operator. The main goal here is to obtain a proper manageable analogue of (6.4.13) suitable for the thermodynamic limit analysis.

Bibliography

[1] P. Woit, "*Quantum Theory, Groups and Representations, An Introduction*", Springer, 2017.

[2] J. E. Humphries, "*Introduction to Lie Algebras and Representation Theory*", New York, NY, 188p., 1970.

[3] J. Von Neumann, "*Mathematical Foundations of Quantum Mechanics*", Princeton University Press, 437p., 1955.

[4] S. M. Sergeev, "*A quantization scheme for modular q-difference equations*", Theoritical and Mathematical Physics, Vol. 142, No. 3, pp.500–509, 2005.

[5] R. M. Kashaev & S. M. Sergeev, "*Spectral equations for the modular oscillator*", Reviews in Mathematical Physics, Vol.30, No.7, pp.1-23, 2018.

[6] D. Sepunaru, "*Elements of Real Hilbert Spaces Theory*", Scientific Research Publishing Inc., Vol.2, No. 5, 9p., 2015.

[7] E. W. Weisstein, "*Schrödinger equation*", Wolfram Research Inc., 1p., 1996-2007.

[8] P. Ball, "*Mysterious Quantum Rule Reconstructed From Scratch*", Quanta magazine, 2021.

[9] George W. Mackey, "*Unitary Group Representations: In Physics, Probability, and Number Theory*". Addison-Wesley, 1989.

[10] S. Burris & H.P. Sankappanavar, "*A Course in Universal Algebra*", The Millenium Edition, 2012.

[11] R. M. Kashaev & S. M. Sergeev, "*On the Spectrum of the Local* $\mathbb{P}^2$ *Mirror Curve*", Ann. Henri Poincaré 21, pp. 3479–3497, 2020.

[12] E. T. Whittaker & G. N. Watson, "*A course of Modern Analysis*", Cambridge University Press, 4th edition, rep, 1999.

[13] E. R. Senapathi,"*Generalized Casimir Operators*", Journal of Algebra and its Applications, Vol. 18, No.11, pp. 1-22, 2016.

[14] T. D. Palev, "*Fock space representations of the Lie superalgebra*". Journal of Mathematical Physics, Vol.21, Issue 6, AIP Publishing LLC, 2020.

[15] M. Alaoui & A. Haily, "*The converse of Schur's Lemma in group rings*", Publicacions Matemàtiques, Vol.50, Issue 1, 2006.

[16] J. M. Dahlström, A. l'Huillier & J. Moritsson, "*Quantum mechanical approach to probing the birth of attosecond pulses using a two-color field*", Journal of Physics B: Atomic, Molecular and Optical Physics, IOP Publishing, Vol. 44, No. 9, 2011.

[17] V. Kac, "*Introduction to lie Algebra: Cartan subalgebra. Encyclopedia of Mathematics*", MIT Mathematics, Lectures 8, 2010.

[18] H. M. Srivastava, A. Çetinkaya & İ. O. Kımaz, "*A certain Generalized Pochhammer Symbol and its applications to hypergeometric functions*". Applied Mathematics and Computation, Elsevier, Vol. 226, pp. 484-491, 2014.

[19] G. Bellamy, D. Rogalski, T. Schedler, J. T. Stafford & M. Wemyss, "*Noncommutative Algebraic Geometry*". Mathematical Sciences Research Institute Publications, Cambridge University Press, No. 64, 353p., 2016.

[20] L. Boza, E. M. Fedriani, J. N. Úñez & Á. F. Tenorio, "*A historical review of the classifications of Lie algebras*". Revista de la Union Matemática Argentina, Vol. 54, No. 2, pp.75-99, 2013.

[21] R. N. Cahn, "*Semi-Simple Lie Algebras and Their Representations*". The Benjamin/Cummings Publishing Company. Lawrence Berkeley Laboratory University of California Berkeley, California, 156p., 1984.

[22] R. Borcherds, "*Generalized Kac-Moody Algebras*". Department of Pure Mathematics and Mathematical Statistics. University of Cambridge. Journal of Algebra 115, pp. 501-512, 1988.

[23] Y. Billig, M. Lau, A. Milas & K. Misra, "*Vertex Algebras and Quantum Groups*". Carlton, Laval, SUNY-Albany, and North Carolina State Universities, 17p., 2016.

[24] G. G. de Polavieja, "*A causal quantum theory in phase space*". Physics Letters A, Volume 220, Issue 6, 16 September 1996, pp. 303-314. Elsevier B.V., 2020.

[25] B. Zwiebach, "*Dirac's Bra and Ket Notation*", MIT Course, Quantum Physics II, Lectures Notes, 15p., 2013.

[26] U.S. Department of Energy's NNSA, "*Quantum Notation*". Los Alamos National Security, LLC, 1p., 2010-11.

[27] T. S. Ratiu & O. G. Smolyanov, " *Hamiltonian Structures in the Quantum Theory of Hamilton–Dirac Systems*", Doklady Mathematics, MAIK Nauka/Interperiodica, 91(1), pp. 68-71, 2015.

[28] A. Kojevnikov, "*P.A.M. Dirac*", Encyclopaedia Britannica, Inc. 2020.

[29] A. G. Izergin & V. E. Korepin, "*The quantum inverse scattering method approach to correlation functions.*". Communications in Mathematical Physics. 94, No. 1, pp. 67–92, 1984.

[30] V. E. Korepin, N. M. Bogoliubov & A. G. Izergin, "*The Quantum Inverse Scattering Method*". Cambridge University Press, 20 Dec. 2020.

[31] M. Jimbo, "*Introduction to the Yang-Baxter Equation*", International Journal of Modern Physics A, Vol. 04, No. 15, pp. 3759-3777, 1989.

[32] E. C. Rowell, "*From Quantum Groups to Unitary Modular Tensor Categories*", Contemporary Mathematics 2006, Academia, pp. 1-14, 2021.

[33] S. M. Khoroshkin & V. N. Tolstoy,"*Universal R-Matrix for quantum supergroups*", Lecture Notes Physics, pp. 457-488, 1991.

[34] E. Corrigan, D. B. Fairlie & P. Fletcher, "*Some aspects of quantum Groups and supergroups*", Journal of Mathematical Physics, Vol. 31, Issue 4, pp.776-780, 1990.

[35] F. D'Andrea, "*Noncommutative Geometry and Quantum Group Symmetries*", Intenational School for Advanced Studies, Mathematical Physics Sector, 143p., 2006-2007.

[36] L. A. Takhtadzhan & L. D. Faddeev, "*The Quantum Method Of The Inverse Problem And The Heisenbeg XYZ Model*". Uspekhi Mat. Nauk, Vol. 34, No.5(209), pp. 13-63, 1979; Russian Math. Surveys, Vol. 34, No. 5, pp.11-68, 1979.

[37] J. Collins & R. Duncan, "*Hopf-Frobenius algebras and a simpler Drinfeld double*". University of Strathclyde, Department of Computer and Information Sciences, Lecture notes, 26p., 2019.

[38] N. Y. Reshetikhin, L. A. Takhtadzhyan & L. D. Faddeev, "*Quantization of Lie groups and Lie algebras*", Algebra i Analiz, 1:1, pp. 178–206, 1989; Leningrad Mathematical Journal, 1:1, pp. 193–225, 1990.

[39] A. Gray, "*Euclidean Space in Modern Differential Geometry of Curves and Surfaces with Mathematica.*" , 2nd ed., CRC Press, pp. 2-5, 1997.

[40] M. Matone, "*Explicit Baker-Campbel-Hausdorf formulae for the generators of semisimple complex Lie algebras*", European Physics Journal, Vol. 76, No. 610, pp. 1-7, 2016.

[41] R. Zhang, "*The Baker-Campbell-Hausdorff Formula*", Math.columbia.edu, lecture Notes, 8p., 2017.

[42] S. Sternberg, "*Lie algebras*", Harvard Mathematics, Lecture Notes, 196p., 2004.

[43] R. Talley, "*An Introduction to Lie Algebra*", California State University, Thesis Lecture, 54p., 2017.

[44] J. Lang, "*A Jacobian identity in positive characteristic*", Journal Commutative Algebra, Vo. 7, No. 3, pp. 393–409, 2015.

[45] A. Peer, "*Angular Momentum in Quantum Mechanics*", University College Cork, Lecture Notes, 24p., 2018.

[46] A. Neumaier & D. Westra,"*Classical and quantum mechanics via Lie algebras*", University of Vienna, 503p., 2011.

[47] P. Senesi, "*Finite-dimensional representation theory of loop algebras: A survey*", University of Ottawa, Department of Mathematics and Statistics, Lecture Notes, 25p., 2009.

[48] A. Kempf, "*Advanced Quantum Theory*", University of Waterloo, Canada, Department of Applied Mathematics, Lecture Notes, 78p. 2016.

[49] V. V. Bazhanov & S. M. Sergeev, "*Yang-Baxter Maps, Discrete Integrable Equations and Quantum Groups*", Cornell University, Mathematical Physics, 36p., 2012.

[50] P. Etingof, V. Ginzburg, N. Guay, D. Hernandez & A. Savage, "*Twenty-five years of representation theory of quantum groups*", Banff International Research Station (BIRS), Lecture Notes, 12p., 2011.

[51] E. W. Weisstein, "*Pauli Matrices*", From MathWorld–A Wolfram Web Resource, Wolfram Research, Inc., 1999-2021.

[52] P. J. Olver, "*Noether's Two Theorems*", University of Minnesota, Convergence, Perimeter Institute, 2015.

[53] M. Bañados & I. Reyes, "*A short review on Noether's theorems, gauge symmetries and boundary terms*, Faculdad de Física, Pontificia Universidad Católica de Chile, pp.5-66, 2017.

[54] J. F. Pommaret, "*Bianchi identities for the Riemann and Weyl tensors*", HAL, hal-01289025f, 34p., 2016.

[55] M. Von Steinkirch, "*Introduction to Group Theory for Physicists*", State University of New York, 108p., 2011.

[56] J. Malkoun & N. Nahlus, "*Commutators and Cartan subalgebras in Lie algebras of compact semisimple Lie groups* ", Journal of Lie Theory, Vol.27, No. 4, pp. 1-19, 2016.

[57] S. Bouarraoudj, P. Grozman & D. Leites, "*Classification of Finite Dimensional Modular Lie Superalgebras with Indecomposable Cartan Matrix?*". Symmetry, Integrability and Geometry: Methods and Applications. SIGMA 5, 060, 63p., 2009.

[58] C-S. Chu, "*Cartan-Weyl 3-algebras and the BLG Theory I: Classification of Cartan-Weyl 3-algebras*", Journal of High Energy Physics, Vol. 2010, Issue 10, No. 50, pp. 1-29, 2010.

[59] V. Kac, "*Introduction to Lie Algebras — Cartan Matrices and Dynkin Diagrams*", MIT Math, Lecture_17_Notes, 7p., 2010.

[60] C. Perez & D. Rivera, "*Serre type relations for complex semisimple Lie algebras associated to positive definite quasi-Cartan matrices*", Elsevier Inc., Linear Algebra and Its Applications, Vol. 567, pp. 14-44, 2019.

[61] J. Mostowski & J. Pietraszewicz, "*Quantum versus classical angular momentum*", European Journal of Physics, IOP Publishing, Vol. 41, No 2, 13p., 2020.

[62] B. Zwiebach, "*Spin one-half, Bras, Kets and operators*", MIT OpenCourseWare, Lecture 4, 18p., 2013.

[63] F. M. Alamrani, "*On eigenstates for some sl_2 related Hamiltonian*", Acta Universitatis Apulensis 57, pp. 93-100, doi: 10.17114/j.aua.2019.57.08, 2019.

[64] K. A. Brown & K. R. Goodearl, " *Lectures on Algebraic Quantum Groups*", Springer AG Basel, p. 9, 2002.

[65] S. M. Sergeev, "*Symmetries, Groups and Algebras in Physics*", Lecture notes, Australian National University, Faculty of Science, pp. 2-54, 2007

[66] F. M. Alamrani, "*Representation theory of sl_2, $Uq(sl_2)$, oscillator, and q-oscillator*", University of Canberra, under preparation, 2021.

[67] Downie, Ryan W., "*An introduction to the theory of quantum groups*" (2012). EWU Masters Thesis Collection. 36.